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Topologia de Grothendieck em geometria algébrica e geometria algébrica real

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Topologia de Grothendieck em geometria algébrica e geometria algébrica real

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Em homenagem ao falecimento de Alexander Grothendieck,
completando 10 anos.

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por todo o apoio e orientação que me ofereceram ao longo dessa jornada.

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"Wir müssen wissen.
Wir werden wissen."
— David Hilbert.

Resumo

Chen, Haoyu **Topologia de Grothendieck em geometria algébrica e geometria algébrica real**. 2024. 135 p. Monografia (Bacharelado em Matemática) – Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2º Semestre de 2024.

A noção de topologia de Grothendieck, introduzida por Alexander Grothendieck juntamente com a topologia étale, visa definir feixes em categorias arbitrárias e suas cohomologias, como parte da tentativa de provar a conjectura de Weil, que acabou sendo bem-sucedida. Neste trabalho, exploramos as topologias de Grothendieck na categoria de morfismos étale em um esquema X, incluindo a topologia étale e a topologia real étale. Estudamos também que a categoria de feixes na topologia real étale de um esquema X é equivalente à categoria de feixes em espectro real associado ao esquema X. Por fim, discutimos a relação entre a cohomologia de Galois e a cohomologia étale sobre um corpo, além das interações entre pontos reais, ordens e dimensão cohomológica.

Palavras-chave: feixes, topologias de Grothendieck, geometria algébrica, geometria algébrica real e cohomologia de Galois

Abstract

CHEN, HAOYU **Grothendieck topologies in algebraic geometry and real algebraic geometry**. 2024. 135 p. Monografia (Bacharelado em Matemática) – Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2º Semestre de 2024.

The notion of Grothendieck topology, introduced by Alexander Grothendieck together with the étale topology, aims to define sheaves on arbitrary categories and their cohomologies, as part of the attempt to prove Weil's conjecture, which ended up being successful. In this work, we explore the Grothendieck topologies in the category of étale morphisms on a scheme X, including the étale topology and the real étale topology. We also study that the category of sheaves on the real étale topology of a scheme X is equivalent to the category of sheaves on the real spectrum associated to the scheme X. Finally, we discuss the relationship between Galois cohomology and étale cohomology over a field, in addition to the interactions between real points, orders and cohomological dimension.

Keywords: sheaves, Grothendieck topologies, algebraic geometry, real algebraic geometry and Galois cohomology

Lista de Abreviaturas e Siglas

et	Topologia étale (*Étale Topology*)
ret	Topologia real étale (*Real Étale Topology*)
SGA	Séminaire de Géométrie Algébrique du Bois Marie
PID	Domínio principal (*Principal Ideal Domain*)
gcd	Máximo divisor comum (*greatest Common Divisor*)
lcm	Mínimo múltiplo comum((Least Common Multiple))
URL	Localizador Uniforme de Recursos (*Uniform Resource Locator*)
IME	Instituto de Matemática e Estatística
USP	Universidade de São Paulo

Lista de Símbolos

Set	Category of sets
Ab	Category of abelian groups
Gr	Category of groups
Ring	Category of Rings
Scheme	Category of Schemes
$Psh(X)$	Category of presheaves on X
$Sh(X_\tau)$	Category of sheaves of sets on X_τ
$Ab(X_\tau)$	Category of sheaves of abelian groups on X_τ
$\prod_i A_i$	Product
$\coprod_i A_i$	Coproduct
$\varprojlim$	Projective limit
$\varinjlim$	Injective limit
h_U	Hom-functor $hom(-,U)$
$\emptyset$	Initial object of category
$*$	Final object of category
0	Zero object of category
$M \oplus N$	Direct sum (or Biproduct) of M and N
$\mathrm{Ker}(f)$	The object of Kernel of f
$\ker(f)$	The morphism of Kernel of f
$\mathrm{CoKer}(f)$	The object of cokernel of f
$\mathrm{coker}(f)$	The morphism of cokernel of f
$\mathrm{Im}(f)$	The object of image of f
$\mathrm{im}(f)$	The morphism of image of f
$\mathrm{Coim}(f)$	The object of coimage of f
$\mathrm{coim}(f)$	The morphism of coimage of f
$M^*(A)$	Injective resolution of A
R^iF	n-th right derived functor of F
$H^i(-)$	i-th cohomology group functor
$\Gamma(X,F)$	Section of F on X
a_τ	Sheafification functor with respect to τ
X_{et}	Étale site
X_{ret}	Real étale site

$\mathcal{C}^{op}$	Dual category of $\mathcal{C}$
$k[X_1, ..., X_n]$	Polynomial ring in n indeterminates over k
$\mathbb{Z}[\sqrt{-1}]$	The group ring generated by $\sqrt{-1}$
$\mathbb{A}_k^n$	Set of algebraic number in k^n
$V(I)$	Zero locus of I
$I(V)$	Vanishing ideal of the set V
$\sqrt{I}$	Radical of ideal I
$M \otimes_A N$	Tensor product of M and N over A
$k[V]$	Coordinate ring of V
$k(V)$	Rational function field of V
$Frac(A)$	Field of fractions of A
$Spec\ A$	Zariski spectrum of A
$S^{-1}A$	The localization of A by S
A_f	The localization of A by $\{1, f, f^2, ..\}$
$A_\mathfrak{p}$	The localization $(A \setminus \mathfrak{p})^{-1}(A)$
$\mathfrak{p}A_\mathfrak{p}$	The tensor product $\mathfrak{p} \otimes_A A_\mathfrak{p}$
$\mathcal{O}_X$	Structure sheaf of a scheme X
$\mathcal{O}_{X,x}$	Fiber of $\mathcal{O}_X$ at x
$\kappa(x)$	Quotient field of point x
K/F	K is a field extension of F
$Gal(K/F)$	Galois group associated to field extension K/F
A^H	H-invariant set of A
$A^\times$	Group of invertible elements of A
$\models$	Models of
$\sqrt[R]{I}$	Real radical of ideal I
$Sper\ A$	Real spectrum of A
$\kappa(\xi)$	Real closure of $\kappa(supp(\xi))$ with respect to ξ
$\rightsquigarrow$	Specialization relation
X_r	Real spectrum of a scheme X
$cd(X)$	Cohomological dimension of X
$cd_p(X)$	Cohomological p-dimension of X

Contents

Preface

Algebraic geometry is a branch of mathematics that traces its origins back to ancient civilizations, where geometric problems were often framed in terms of algebraic equations and vice versa. Initially, the focus was on the vanishing sets of collections of polynomials with coefficients in fields such as $\mathbb{C}$, $\mathbb{R}$, $\mathbb{C}$ or finite field $\mathbb{F}_q$. The primary aim was to study the geometry of these vanishing sets rather than to solve polynomial equations explicitly. This involves concepts such as dimension, degree, genus, irreducibility, regularity, and normality in the context of these varieties.

Between the 19th and 20th centuries, the field of algebraic geometry evolved to focus on algebraic varieties, i.e. spaces that can be locally identified with the vanishing sets of polynomial collections, such as projective varieties. This change led to significant results such as Bézout's theorem, the Riemann-Roch theorem, the GAGA principle, and Chow's lemma. As a result, algebraic geometry has become a vibrant intersection of many branches of mathematics.

A significant milestone in algebraic geometry was Hilbert's Nullstellensatz, proven by David Hilbert at the end of the 19th century. This theorem established a crucial link between the common zero sets of polynomial systems and the ideals of polynomial rings, allowing geometric properties to be explored through pure algebra. It also demonstrated that for any algebraically closed field k, the category of irreducible algebraic sets over k is contravariantly equivalent to the category of finitely generated k-algebras.

In the early 20th century, significant advancements were made in various areas of mathematics, particularly in topology. Henri Poincaré introduced the field in his work "Analysis Situs" (1895), which laid the groundwork for concepts such as fundamental groups and singular homology. In the latter half of the century, mathematicians like Veblen, Alexander, Lefschetz, and Čech developed various methods for associating (co)homology groups with topological spaces. This progress was further clarified by Eilenberg and Steenrod, who demonstrated that any (co)homology theory constructed in a concrete and purposeful manner on topological spaces was equivalent, provided it satisfied a short list of axioms known as the Eilenberg-Steenrod axioms.

In the 1940s, Leray sought to understand the relationship between the cohomology groups of two spaces X and Y connected by a continuous map from Y to X. This exploration led to the introduction of sheaves, which are local systems of coefficient groups, as well as the development of sheaf cohomology and spectral sequences. Later, in his influential 1957 Tohoku paper ([13]), Grothendieck demonstrated that the category of sheaves of abelian groups on a topological space forms an abelian category with enough injectives. This allowed for the definition of the cohomology groups of sheaves on a space X as the right derived functors of the functor that assigns to each sheaf its abelian group of global sections, which is a left exact functor.

To develop homological algebra, S. Mac Lane and S. Eilenberg introduced category theory, focusing on natural transformations and functors. In this framework, Grothendieck and Serre's school reformulated algebraic geometry using the language of category theory, commutative algebra, sheaf theory, and its cohomology. They introduced the concept of schemes in place of algebraic varieties, allowing classical algebraic geometry to be expressed through schemes.

One key advantage of schemes is that they do not require a priori embedding in affine or projective spaces; they can be realized geometrically with coordinates in any field or ring. This flexibility means that base change is a natural operation for schemes. Additionally, schemes are topological spaces equipped with sheaves, enabling the use of sheaf cohomology as a powerful tool in the study of their properties.

However, this foundation proved inadequate for certain situations, such as establishing a cohomology theory necessary for the Weil conjectures. The Zariski topology is quite coarse, leading to the result that for any irreducible topological space X, the cohomology group $H^n(X, F) = 0$ for all constant sheaves and all $n > 0$. A revolutionary solution to this problem was introduced by the concept of a site or Grothendieck topology. This topology does not follow the usual sense of topology; rather, it generalizes the notion of an open cover to arbitrary categories. For example, in the étale topology, not only are open subschemes is considered open, but any étale morphism $U \to X$ is also regarded as an open subset. This enriched perspective on the Zariski topology allowed for the development of a more robust cohomological framework, effectively addressing the requirements of the Weil conjectures.

The real spectrum emerged from the work of Coste and Coste-Roy in the early 1980s, as they investigated the topos of real étale sheaves $Sh(\text{spec}A_{ret})$ for a commutative ring with unit A ([9]). Their motivating question was whether this topos is spatial; that is, whether there exists a category of sheaves on some topological space that is equivalent to the category of sheaves on the real étale site. The answer is affirmative, and the topological space that satisfies this condition is known as the real spectrum of A.

Later, Claus Scheiderer demonstrated that for any scheme X, the category of sheaves on the real spectrum X_r is naturally equivalent to the category of sheaves on the real étale topology of Et/X. This finding encouraged further exploration of the relationship between real étale topology and étale topology. While neither topology was finer than the other, Scheiderer proposed a comparison through an intermediate topology b, defined as the intersection of both topologies. This approach proved successful, as the category of sheaves on b is equivalent to the glued category of $Sh(X_{et})$ and $Sh(X_{ret})$. Scheiderer utilized this result to uncover the deep connections that exist between the étale site of a scheme X and the orderings of the residue fields of X.

This work aims to study various Grothendieck topologies on the category of étale morphisms into a scheme X such as étale topology and real étale topology. In this thesis, a ring refers to a commutative ring with 1. We assume that the reader possesses a solid understanding of commutative algebra, category theory, and algebraic geometry. In the first chapter, we will summarize the fundamental notions and results that will be referenced throughout the thesis, without delving into extensive details on these subjects, due to the limited time available to address the "unbounded" mathematical world. The second chapter focuses on the main object of the thesis, the étale site, where we define the étale topology and provide some classical examples of étale sheaves. The third chapter offers a brief overview of real algebraic geometry, covering topics such as real closed fields, the real Nullstellensatz, real valuation rings, and the real spectrum. In chapter four, we present the specialization in the real spectrum of schemes, which serves as an important tool for our next discussion, the reader may skip this chapter if they are willing to accept the ultimate results presented in later sections. In chapter five, we introduce another Grothendieck topology, the real étale site X_{ret}, and prove that the real étale topos, the category of sheaves on this site, is spatial. Chapter six explores the "gluing" of the étale topos and the real étale topos, a concept utilized by Claus Scheiderer in his work on real and étale cohomology. Finally, in the last chapter, we discuss the relationship between Galois cohomology and étale cohomology on a spectrum of fields, as well as the interplay among real points, orderings, and cohomological dimensions.

Chapter 1

Prerequisites

We summarize the fundamental notions and results that will be referenced throughout the thesis, without delving into extensive details on these subjects. We will recommend some valuable references at the beginning of each section for those who want to gain a deeper understanding of these subjects.

1.1 Algebraic Geometry

In this section, we will not delve into the details of classical algebraic geometry; instead, we will cover only the basic concepts necessary to introduce modern algebraic geometry. The main references for this part are [15] (for classic algebraic geometry), [14], [29], and [17].

Definition 1.1.1. *Let k be a field.*

i - If $S \subseteq k[X_1, ..., X_n]$, then we define

$$V(S) := \{x \in \mathbb{A}^n_K : p(x) = 0, \forall p \in S\}$$

ii - If $Y \subseteq \mathbb{A}^n_K$, then we define

$$I(Y) = \{f \in k[X_1, ..., X_n] : f(x) = 0,\ \forall x \in Y\}$$

*iii - A subset $X \subseteq \mathbb{A}^n_K$ is an **algebraic set** in $\mathbb{A}^n$ if there exists $S \subseteq k[X_1, ..., X_n]$ such that $X = V(S)$.*

Lemma 1.1.2. *i - If $X \subseteq \mathbb{A}^n_K$, then $I(X)$ is a radical ideal.*

ii - If $I, J \subseteq k[X_1, ..., X_n]$ are two subsets such that $I \subseteq J$, then $V(J) \subseteq V(I)$.

iii - If X and Y are algebraic sets such that $X \subseteq Y \subseteq \mathbb{A}^n$, then $I(Y) \subseteq I(X)$.

iv - If X is an algebraic set, then $X = V(I(X))$.

v - If $X, Y \subseteq \mathbb{A}^n_K$ are algebraic sets, then $X \cup Y = V(I(X) \cap I(Y))$ and $X \cap Y = V(I(X) + I(Y))$.

vi - $\emptyset$ and $\mathbb{A}^n_K$ are algebraic sets.

Using the last two items, we can define a closed topology on $\mathbb{A}^n_K$, known as the **Zariski topology**, in which the closed subsets are algebraic sets.

Definition 1.1.3. *Let $V \subseteq \mathbb{A}^n_K$ be an algebraic set,*

- *V is said to be **reducible** if and only if there exists two closed subsets $V_1, V_2 \subsetneq V$ in subspace topology (induced by Zariski topology) such that $V_1 \cup V_2 = V$.*
- *V is said to be **irreducible** if it is not reducible.*

This definition is a specific case of an irreducible topological space.

Theorem 1.1.4 (Hilbert's Nullstellensatz)**.** *Let K be an algebraically closed field, and let $A = k[X_1, ..., X_n]$, then,*

a - An algebraic set $X \subseteq \mathbb{A}^n_K$ is irreducible if and only if $I(X)$ is a prime ideal. (It is not necessary to assume that the field is algebraically closed.)

b - $m \subseteq A$ is a maximal ideal if and only if m is of the form

$$m = (X_1 - a_1, ..., X_n - a_n) = I(P)$$

for some $P = (a_1, ..., a_n) \in \mathbb{A}^n_K$.

c - If $J \subsetneq A$ ideal, then $V(J) \neq \emptyset$.

d - If $J \subsetneq A$ ideal, then

$$I(V(J)) = \sqrt{J}$$

.

Proof. the proof of item a) can be found in Proposition 1.8 of [15], and b), c), and d) in Theorem 1.15 of [15]. □

Corollary 1.1.5. *Let k be an algebraically closed field, and let $A = k[X_1, ..., X_n]$. The maps V : $\{ideals\ of\ A\} \to \{subsets\ of\ \mathbb{A}^n_k\}$, $I : \{subsets\ of\ \mathbb{A}^n_k\} \to \{ideals\ of\ A\}$ induce the bijective functions between*

- $\{algebraic\ sets\ in\ \mathbb{A}_k^n\}$ *and* $\{radical\ ideals\ of\ A\}$;
- $\{irreducible\ algebraic\ sets\ in\ \mathbb{A}_k^n\}$ *and* $\{prime\ ideals\ of\ A\}$;
- $\{points\ in\ \mathbb{A}_k^n\}$ *and* $\{maximal\ ideals\ of\ A\}$.

Definition 1.1.6. *A* ***polynomial function*** *on an algebraic set* $V \subseteq \mathbb{A}_k^n$ *is a map* $f : V \to k$ *such that there exists a polynomial* $p \in k[X_1, ..., X_n]$ *with* $f(x) = p(x)$ *for all* $x \in V$.

Definition 1.1.7. *Let* $V \subseteq$ *be an algebraic set. The* ***coordinate ring*** *of* V *is defined by*

$$k[V] := k[X_1, ..., X_n]/I(V).$$

Since for any p and $q \in k[X_1, ..., X_n]$ we have

$$p|_V = q|_V \iff (p-q)|_V = 0 \iff p - q \in I(V).$$

For this reason, the coordinate ring of V can be identified as

$$k[V] = \{f : f : V \to k \text{ is a polynomial function}\}$$

Let $Irr(k)$ be a category whose objects are affine varieties in k^n, for some n. For any affine varieties $X \subseteq k^n$, $Y \subseteq k^m$, a morphism $\phi : X \to Y$ is a map that can be expressed by polynomials in the coordinate, i.e., there exist polynomials $f_1, ..., f_m \in k[X_1, ..., X_n]$ such that for each point $(a_1, ..., a_n) \in X$,

$$\phi(a_1, ..., a_n) = (f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n))$$

(this map is also called a polynomial map).

Now, let $f : X \to Y$ be a morphism between affine varieties. For $g \in k[Y]$ we define $f^*(g) := g \circ f$. Since g is a polynomial function. Thus we have a map between k-algebras

$$\begin{aligned} f^* : k[Y] &\longrightarrow k[X] \\ g &\longmapsto f^*(g) = g \circ f. \end{aligned}$$

Moreover, this map is a homomorphism of the k-algebra, since

$$f^*(g_1 + g_2) = (g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f = f^*(g_1) + f^*(g_2),$$

$$f^*(g_1 \cdot g_2) = (g_1 \cdot g_2) \circ f = (g_1 \circ f) \cdot (g_2 \circ f) = f^*(g_1) \cdot f^*(g_2)$$

and for any constant $c \in k$, it is clear that $f^*(c) = c$.

Since

$$(g \circ f)^*(h) = h \circ (g \circ f) = (h \circ g) \circ f = g^*(h) \circ f = f^*(g^*(h)),$$

and since

$$(id_X)^*(f) = f \circ id_V = f = id_{k[X]},$$

the map

$$V \longmapsto k[V]$$
$$f : V \to W \longmapsto f^* : k[W] \to k[V]$$

defines a contravariant functor between category $Irr(k)$ and the category of finitely generated k-algebras which are integral domains. In addition,

Theorem 1.1.8. *The functor above is a contravariant equivalences of categories.*

Therefore, to study algebraic sets, it is sufficient to focus on the prime ideals of associated polynomial rings.

Since concepts such as singularity, normality, and projective algebraic sets will not be used in our work, we will not introduce them here.

Now, we will review some basic notions of sheaves on topological spaces, which will be used to define schemes, a generalization of algebraic varieties.

For any topological space X, we define a category $O(X)$, whose objects are the open subsets of X, and the only morphisms are the inclusion maps.

Definition 1.1.9. *Let X be a topological space. A **presheaf** F of sets (resp. rings, abelian groups, groups, etc.) in X is a contravariant functor $F : O(X) \to$ **Set**.*

If F is a presheaf on X, the set $F(U)$ is called the **sections** of the presheaf F on the open set U, and sometimes we use the notation $\Gamma(U, F)$ instead of $F(U)$. We call the maps $F(V \hookrightarrow U)$ by **restriction** maps, and we sometimes use $s|_V$ instead of $F(V \hookrightarrow U)(s)$, for $s \in F(U)$.

Definition 1.1.10. *A presheaf F on a topological space X is a **sheaf** if it satisfies the following axioms:*

1. *If U is an open subset of X, if $\{U_i\}_{i \in I}$ is an open covering of U, and if $s, s' \in F(U)$ is an element such that $s|_{U_i} = s'|_{U_i}$ for all $i \in I$, then $s = s'$.*

2. *If U is an open set, if $\{U_i\}_{i \in I}$ is an open covering of U, and if we have elements $s_i \in F(U_i)$ for each*

$i \in I$ such that for every $i, j \in I$, the equality

$$s_i|_{U_i \cap U_j} = s_j|_{U_j \cap U_i}$$

holds, then there is an element $s \in F(U)$ such that $s|_{U_i} = s_i$ for every $i \in I$.

Example 1.1.11. *Let V be an irreducible projective curve. Since V is a Riemann surface (see [22]), holomorphic functions can be defined on any open subset of the topology that gives V its Riemann surface structure. The functor given by*

$$\begin{aligned} \mathcal{H} : O(V) &\longrightarrow \textbf{Set} \\ U &\longmapsto \{\text{holomorphic function on } U\} \end{aligned}$$

is a sheaf on V. Moreover, since V is compact, the maximum modulus theorem implies that $\mathcal{H}(V) \cong \mathbb{C}$ (see Theorem 1.37 of [22]).

Example 1.1.12. *Let $V \subseteq \mathbb{C}^n$ be an algebraic set equipped with Zariski topology, the functor given by*

$$\begin{aligned} \mathcal{O} : O(V) &\longrightarrow \textbf{Ring} \\ U &\longmapsto \{\frac{p}{q} : p, q \in \mathcal{C}[V],\ q(x) \neq 0\ \forall x \in U\} \end{aligned}$$

is a sheaf on V. This sheaf is know as structure sheaf of V or regular function sheaf, This name reflects its role in capturing the local properties of algebraic set V, such as dimension and singularity.

If V is an irreducible projective curve, we have $\mathcal{O}(V) \cong \mathbb{C}$ (see Theorem 2.35 of [15]) which coincides with $\mathcal{H}(V)$.

Now, we will introduce the **germ** of the sections of a presheaf at some point.

Definition 1.1.13. *If F is a presheaf on X, and x is a point of X, we define the **fiber (stalk)** F_x of F at x to be the direct (injective) limit*

$$\varinjlim_{\substack{U \in O(X) \\ x \in U}} F(U).$$

Definition 1.1.14. *A **morphism of sheaves (presheaves)** is a natural transformation between contravariant functors. So, an **isomorphism** of sheaves (presheaves) is a morphism $\phi : F \to G$ which has two-side inverse, i.e., exists a morphism of sheaves $\psi : G \to F$ such that $\phi \circ \psi = id_G$ and $\psi \circ \phi = id_F$.*

Proposition 1.1.15. *Let $\phi : F \to G$ be a morphism of sheaves on a topological space X. Then ϕ is an isomomorphism if and only if the induced map on the fiber $\phi_p : F_p \to G_p$ is an isomorphism (which depends on the type of sheaves) for every $P \in X$.*

Proof. See Proposition 1.1, II of [14] □

Given a presheaf F on a topological space X, we can construct the sheaf $F^{\#}$ as follows. For any open set U, let $F^{\#}(U)$ be the set of functions $s : U \to \bigcup_{x \in U} F_x$ such that

- For any $x \in U$, $s(x) \in F_x$;
- For any $x \in U$, there is a neighborhood V of x, contained in U, and an element $t \in F(V)$, such that for every $y \in V$, the germ t_y of t at y coincides with $s(y)$.

The sheaf $F^{\#}$ is called the sheaf associated to the presheaf F. This construction define a functor $(\quad)^{\#} : F \mapsto F^{\#}$ from the category of presheaves X to the category of sheaves on X, called sheafification. Moreover, the sheafification is the left adjoint functor of inclusion functor from category of sheaves on X to the category of presheaves on X.

Definition 1.1.16. *Let $f : X \to Y$ be a continuous map of topological spaces. Given a sheaf F on X, we define the **direct image** sheaf f_*F on Y by $(f_*F)(V) = F(f^{-1}(V))$ for every open subset $V \subseteq Y$. Given a sheaf G on Y, we define the **inverse image** sheaf f^*G on X by the sheaf associated to the presheaf*

$$U \mapsto \varinjlim_{\substack{V \in O(Y) \\ V \supseteq f(U)}} G(V)$$

for every open subset $U \supset X$.

Both constructions define functors and, moreover, form an adjoint pair.

Definition 1.1.17. *Let X be a topological space, and let Z be a subspace of X (i.e., a subset equipped with induced topology). Let $i : Z \hookrightarrow X$ be the inclusion map. Given a sheaf F on X, we define the **restriction of** F to Z by i^*F, we often denote it by $F|_Z$.*

Hilbert-Nullstellensatz theorem established bijective correspondence between prime ideals and irreducible algebraic sets, so instead of working on algebraic sets, we can study **Zariski spectrum** *Spec*. Let's define a topology on its spectrum.

Definition 1.1.18. *Let A be a ring, and let $\mathfrak{a} \subseteq A$ be an ideal, we define the subset $V(\mathfrak{a}) \subseteq SpecA$ to be the set of all prime ideals of A that contain $\mathfrak{a}$.*

Lemma 1.1.19. *Let A be a ring,*

a - If a and b are two ideals of A, then $V(ab) = V(a) \cup V(b)$.

b - If $\{a_i\}$ is a family of ideals of A, then $V(\sum a_i) = \bigcap V(a_i)$.

c - If a and b are two ideals of A, $V(a) \subseteq V(b)$ if and only if $\sqrt{b} \subseteq \sqrt{a}$

This lemma allows us to define a closed topology on $Spec\ A$ in which the closed subsets are of the form $V(a)$. We observe that this topology "coincides" with the Zariski topology on algebraic set (in a certain sense), for this reason, this topology is also called **Zariski's Topology**.

For any element $f \in A$, we denote by $D(f)$ the open complement of $V((f))$. One can easily show that open subsets of the form $D(f)$ form a base for the topology of $Spec\ A$.

Now, let us define a ring sheaf (sheaf whose codomain is the category **Ring**) $\mathcal{O}$ on $SpecA$ that is an analogue to the regular function sheaf. For each prime ideal $\mathfrak{p} \subseteq A$, let $A_\mathfrak{p}$ be the localization of A in $\mathfrak{p}$. For an open $U \subseteq Spec\ A$, we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod_{\mathfrak{p}\in U} A_\mathfrak{p} = \bigcup_{\mathfrak{p}\in U}\{(x, A_\mathfrak{p}) : x \in A_\mathfrak{p}\}$, such that $s(p) \in A_p$ for each p, and each $\mathfrak{p}$, there exists a neighborhood V of $\mathfrak{p}$ that is contained in U, and elements $a, f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $A_\mathfrak{q}$.

Of course, the sums and products of functions from $\mathcal{O}(U)$ are functions from $\mathcal{O}(U)$, and identity always is in each A_p. Therefore, $\mathcal{O}(U)$ is a commutative ring with unity. If $V \subseteq U$ are two open rings, the natural restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ is a ring homomorphism, which shows that $\mathcal{O}$ is a presheaf. Finally, the last condition of the definition of $\mathcal{O}$ guarantees that it is a sheaf. This sheaf is called **a spectrum or structure sheaf** on $Spec\ A$.

Proposition 1.1.20. *Let A be a ring, and let $\mathcal{O}$ be a spectrum sheaf on $Spec\ A$*

1. *For all $\mathfrak{p} \in Spec\ A$, the stalk $O_\mathfrak{p}$ is isomorphic to the local ring $A_\mathfrak{p}$*
2. *$\mathcal{O}(Spec\ A) \cong A$.*

Proof. See Proposition 3.1, 3.2 of [17]. □

It is well known that the localization $k[V]_\mathfrak{p}$ of a coordinate ring $k[V]$ provides local information about the associated algebraic set. For example, the tangent space can be defined as the dual space of k-module m_x/m_x^2, where m_x is the maximal ideal of the localization $k[V]$ by the prime ideal $\mathfrak{p} := \{f \in k[V] : f(x) = 0\}$. This proposition ensures that we can study the local properties of an algebraic set through $Spec\ k[V]$, as well as its topological properties.

A natural way to approach geometry is to study additional structures on a topology. For example, Riemann surfaces and the holomorphic functions defined on them.

Definition 1.1.21. ***A ringed space*** *is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a ring sheaf on X. A ringed space morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ of a continuous function $f : X \to Y$ and a natural transformation $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O} = \mathcal{O}_X \circ f^{-1}$.*

A ringed space $(X, \mathcal{O}_X)$ is a ***locally ringed space*** *if and only if for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. A morphism of locally ringed spaces is a ringed space morphism $(f, f^\sharp)$ such that for every point $x \in X$, $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring homomorphism.*

A locally ringed space isomorphism is a morphism whose components are invertible, i.e., f is a homeomorphism and $f^\sharp$ is a natural isomorphism.

The following result motivates the definition of a scheme.

Theorem 1.1.22. *1. If A is a ring, then $(Spec(A), \mathcal{O})$ is a locally ringed space.*

2. If $\phi : A \to B$ is a ring homomorphism, then ϕ induces a locally ringed space morphism

$$(f, f^\sharp) : (Spec\ B, \mathcal{O}_{Spec\ B}) \to (Spec\ A, \mathcal{O}_{Spec\ A}).$$

3. If A and B are rings, then any locally ringed space morphism from $Spec\ B$ to $Spec\ A$ is induced by a ring homomorphism $\phi : A \to B$ in (2).

Proof. See Proposition 3.14, 2 of [17]. □

Definition 1.1.23. ***An affine scheme*** *is a locally ringed space that is isomorphic to $(Spec\ A, \mathcal{O})$ for some ring A.* ***A scheme*** *is a locally ringed space $(X, \mathcal{O}_X)$ such that for every point it has an open neighborhood $U \subseteq X$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. A morphism of schemes is a morphism of a locally ringed space, analogously to isomorphism. We called X the underlying topological space of scheme $(X, \mathcal{O}_X)$, and $\mathcal{O}_X$ its structure sheaf.*

There are some well-known facts about schemes.

Proposition 1.1.24. *Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset, then $(U, \mathcal{O}_X|_U)$ is a scheme.*

Proof. See 3.9, 2 of [17]. □

Lemma 1.1.25. *Let X, Y be two schemes. We suppose given an open covering $\{U_i\}_{i \in I}$ of Y and the morphism $f_i : Y \to X$ of schemes such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every $i, j \in I$. Then there exists a unique morphism $f : Y \to X$ such that $f|_{U_i} = f_i$.*

Proposition 1.1.26. *If Y is an affine scheme, then for any scheme X, the canonical map*

$$\begin{aligned} \rho_{X,Y} : hom_{Sch}(X, Y) &\longrightarrow hom_{CRings}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \\ (f, f^\#) &\longmapsto f^\#(Y) : \mathcal{O}_Y(Y) \to f_*\mathcal{O}_X(Y) = \mathcal{O}_X(X) \end{aligned}$$

is a bijection and "functorial" in X in the sense that for any morphism of schemes $h : Z \to X$, we have $\rho_{Z,Y} \circ hom_{Sch}(h, Y) = hom_{CRings}(\mathcal{O}_Y(Y), g^{\#}(X)) \circ \rho_{X,Y}$

Proof. See Proposition 3.25, 2 of [17]. □

Now, we will define some special schemes.

Definition 1.1.27.
1. *A scheme X is **quasi-compact** or compact if its underlying topological space is quasi-compact.*
2. *A scheme X is **connected** if its underlying topological space is connected.*
3. *A scheme X is **irreducible** if its underlying topological space is irreducible.*
4. *A scheme X is **reduced** if for every open subset U, the ring $\mathcal{O}_X(U)$ is **reduced**, i.e., no nilpotent elements.*
5. *A scheme X is **integral** if for every open subset U, the ring $\mathcal{O}_X(U)$ is an integral domain.*
6. *A scheme X is **locally noetherian** if it has an open cover by affine subschemes $(Spec\ A_i)_{i\in I}$, where each A_i is a noetherian ring.*
7. *A scheme X is **noetherian** if it is locally noetherian and quasi-compact (compact but not necessarily Hausdorff space). Equivalently, X is noetherian if it can be covered by a finite number of open affine subsets $Spec\ A_i$ with A_i noetherian.*

Proposition 1.1.28. *Every affine scheme is quasi-compact.*

Proof. Let $X = Spec\ A$ be an affine scheme, and let $\{D(f)\}_{f\in F}$ be a covering of X consisting of basic open set. Then,

$$X = \bigcup_{f\in F} D(f) = \bigcup_{f\in F} X \setminus V((f)) = X \setminus \bigcap_{f\in F} V((f)).$$

So,

$$\varnothing = \bigcap_{f\in F} V((f)) = V(\sum_{f\in F}(f))$$

which implies $1 \in \sum_{f\in F}(f)$. From the definition of sum of the ideals, there are $f_1, ..., f_n \in F$ and $a_1, ..., a_n \in A$ such that

$$a_1 f_1 + ... + a_n f_n = 1.$$

So, $\varnothing = \bigcap_{i=1,...,n} V((f_i))$, this says that $X = \bigcup_{i=1,...,n} D(f_i)$. □

Corollary 1.1.29. *A scheme X is quasi-compact if and only if X is a finite union of open affine subscheme.*

Proof. If the scheme is quasi-compact then it is obviously a finite union of affine schemes. If X is a finite union of open affine subschemes, then X is a finite union of open quasi-compact subsets, and therefore, X is quasi-compact. □

Proposition 1.1.30. *A scheme X is locally noetherian if and only if for every open affine subscheme $U = Spec\ A$, A is a noetherian ring. In particular, an affine scheme $X = Spec\ A$ is noetherian if and only if A is a noetherian ring.*

Proof. See 3.46, 2 of [17]. □

The Grothendieck's relative point of view is that much of algebraic geometry should be developed for a morphism of schemes $Y \to X$, i.e., we are more interested in properties of Y relative to X rather than the internal geometry of Y.

Definition 1.1.31. *Let X be a fixed scheme. A **scheme over** X or X-**Scheme** is a scheme Y, together with a morphism of schemes $f_Y : Y \to X$. If Y and Z are X-schemes, a morphism of Y to Z is a morphism of scheme $g : Y \to Z$ such that $f_Z \circ g = f_Y$. We will denote the category of all X-schemes by $Sch(X)$.*

In the language of the schemes, a variety is defined in the following way.

Definition 1.1.32. *Let k be a field. An **affine variety** over k is the affine scheme isomorphic to the $(Spec\ A,\ \mathcal{O})$ for some finitely generated algebra A over k. An **algebraic variety** is a k-scheme X such that there exists a finite covering $\{U_1, ..., U_n\}$, where each U_i is an affine open subscheme that is an affine varieties over k.*

The following result shows that the notion of a scheme and an algebraic variety generalize the notion of "variety" in classical sense.

Proposition 1.1.33. *Let k be an algebraically closed field. And let V be an algebraic set over k. The topological space is homeomorphic to the set of closed points of underlying topological space of $Spec\ k[V]$, and its sheaf of regular functions is obtained by restricting the structure sheaf of $Spec\ k[V]$ via this homeomorphism.*

Proof. See Proposition 2.6, II of [14]. In this reference, Hartshorne has established this result for arbitrary varieties in the classical sense, including projective and quasi-affine varieties. □

Here is a powerful lemma used in many results of scheme theory

Lemma 1.1.34. *Let S be a scheme. Let $\{X_i\}_{i \in I}$ be a family of S-schemes. For each $i \neq j$, suppose given an open subscheme $U_{ij} \subseteq X_i$. Suppose also given for each $i \neq j$ an isomorphism of S-schemes $f_{ij} : X_{ij} \to X_{ji}$ such that*

1. $f_{ii} = id_{X_i}$;

2. $f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$;

3. $f_{ik} = f_{jk} \circ f_{ij}$ on $X_{ij} \to X_{ik}$.

Then there exists an S-scheme X, unique up to isomorphism, with open immersions (of S-schemes) $g_i : X_i \to X$ *such that* $g_i = g_j \circ f_{ij}$ *on* X_{ij}*, and that* $X = \bigcup_{i \in I} g_i(X_i)$.

Proof. See Lemma 3.33, 2 of [17] □

We will now introduce the concept of a point in the context of schemes.

Notation 1.1.35. *Let* $\pi_1 : Y \to X$ *and* $\pi_2 : Z \to X$ *be two X-schemes, the set of X-morphism between them,* $hom_X(Z, Y)$*, is also denoted by* $Y(Z)$ *provided there is no confusion (if* $Z = Spec\ A$*, is denoted by* $Y(A)$ *as well).*

Definition 1.1.36. *Let* $\pi : Y \to X$ *be an X-scheme. A* ***section of*** X *is a morphism of X-schemes* $s : X \to Y$ *such that* $\pi \circ s = id_X$*. The set of all sections of Y is exactly the set* $hom_X(X, Y) = Y(X)$ *(here X represents* $id_X : X \to X$*).*

Example 1.1.37. *Let X be a scheme over a field k. Then we can identify* $X(k)$ *with the set of points* $x \in X$ *such that* $\kappa(x) = k$*: Let* $s \in X(k)$ *be a section, and let x be the image of the point of* $Spec\ k$*. Then the homomorphism* $s_x^\#$ *induces a homomorphism of fields* $\kappa(x) \to k$*. Since* $\kappa(x)$ *is a k-algebra,* $\kappa(x) = k$*. Conversely, if* $\kappa(x) = k$*, then there exists a unique section* $Spec\ k \to X$ *(the composition of the canonical morphism* $Spec\ \mathcal{O}_{X,x} \to X$ *and morphism induced by* $\mathcal{O}_{X,x} \to \kappa(x)$*) whose image is x.*

This motivates the following definition:

Definition 1.1.38. *Let X be a scheme over a field k. And let* k' *be any field extension of k. The points of* $X(k')$ *is called k-**rational points** or k-**valued points** of X (here* k' *represents* $k \to k'$*).*

Remark 1.1.39. *The assumption that* k' *is a field extension of k arises from the fact that the k-scheme* $Spec\ k' \to Spec\ k$ *corresponds to a field homomorphism* $k \to k'$*, which is always injective. This injectivity holds because the unique prime ideal of a field is* 0*, implying that the kernel of a field homomorphism must be* 0*.*

As mentioned above, if $k' = k$*, then* $Y(k)$ *is the set of points* $x \in X$ *such that* $\kappa(x) = k$

There are some special morphisms of schemes

Definition 1.1.40. *A morphism* $f : Y \to X$ *of schemes is* ***quasi-compact*** *if X can be covered by open affine subschemes* V_i *such that the pre-images* $f^{-1}(V_i)$ *are quasi-compact.*

Proposition 1.1.41. *If $f : Y \to X$ is a morphism of affine schemes, then f is quasi-compact.*

Proof. $f^{-1}(X) = Y$ is quasi-compact, since Y is an affine scheme. □

Definition 1.1.42. *A homomorphism $\phi : A \to B$ is of* ***finite type*** *if B is a finitely generated (or finite type) A-algebra, i.e., B is isomorphic to $A[x_1, ..., x_n]/I$ for some n and some ideal $I \subseteq A[x_1, ..., x_n]$.*

Definition 1.1.43.

1. *A morphism $f : Y \to X$ of schemes is of* ***finite type at*** *$y \in Y$ if there exists an affine open neighborhood $U = Spec\ B \subseteq Y$ of y and an affine open subscheme $V = Spec\ A \subseteq X$ with $f(U) \subseteq V$ such that the induced ring homomorphism $A \to B$ is of finite type.*

2. *A morphism $f : Y \to X$ of schemes is* ***locally of finite type*** *if it is of finite type at every point of Y. Equivalently, f is locally finite type if X has an open covering by affine subschemes $V_i = Spec\ A_i$ such that for each i, $f^{-1}(V_i) = \bigcup_{j\in J} U_{i,j}$ is covered by open affine subschemes of Y, and the restriction of f to $U_{i,j}$ induces a finite type ring homomorphism $A_i \to B_{i,j}$.*

3. *A morphism $f : Y \to X$ of schemes is of* ***finite type*** *if f is locally of finite presentation and quasi-compact. Equivalently, f is finite type if X has an open covering by affine subschemes $V_i = Spec\ A_i$ such that for each i, $f^{-1}(V_i) = \bigcup_{j\in J} U_{i,j}$ is covered by finitely many open affine subschemes of Y, and the restriction of f to $U_{i,j}$ induces a finite type ring homomorphism $A_i \to B_{i,j}$. In this case, we say X is of* ***finite type over*** *Y*

Definition 1.1.44. *The morphism $f : Y \to X$ is finite if X can be covered by affine open schemes $Spec\ A_i$ such that each $f^{-1}(Spec\ A_i) = Spec\ B_i \subseteq Y$ is affine, and B_i is a finitely generated A_i-module. In this case, we say X is* ***finite over*** *Y.*

Definition 1.1.45. *A homomorphism $\phi : A \to B$ is of* ***finite presentation*** *if B is isomorphic to $A[x_1, ..., x_n]/(f_1, ..., f_m)$ as a finite type A-algebra for some n, m and some polynomials f_i.*

Definition 1.1.46.

1. *A morphism $f : Y \to X$ of schemes is of* ***finite presentation at*** *$y \in Y$ if there exists an affine open neighborhood $U = Spec\ B \subseteq Y$ of y and an affine open subscheme $V = Spec\ A \subseteq X$ with $f(U) \subseteq V$ such that the induced ring homomorphism $A \to B$ is of finite presentation.*

2. *A morphism $f : Y \to X$ of schemes is* ***locally of finite presentation*** *if it is of finite presentation at every point of Y. Equivalently, f is locally finite presented if X has an open cover by affine subschemes $V_i = Spec\ A_i$ such that for each i, $f^{-1}(V_i) = \bigcup_{j\in J} U_{i,j}$ is covered by open affine subschemes of Y, and the restriction of f to $U_{i,j}$ induces a finite presented ring homomorphism $A_i \to B_{i,j}$.*

Definition 1.1.47. *An **open subscheme** of a scheme $(X, \mathcal{O}_X)$ is a scheme $(U, \mathcal{O}|_U)$ where U is an open subspace of X, and $\mathcal{O}|_U$ is the restriction of the structure sheaf of X. An **open immersion** is a morphism $f : X \to Y$ which induces an isomorphism of X with an open subscheme of Y.*

Definition 1.1.48. *A **closed immersion** is a morphism $f : Y \to X$ of schemes such that f induces a homeomorphism of Y onto a closed subset of X, and furthermore the map $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves on X is surjective. A **closed subscheme** of a scheme X is an equivalence class of closed immersions, where we say $f : Y \to X$ and $f' : Z \to X$ are equivalent if and only if there is an isomorphism $i : Y \to Z$ such that $f = f' \circ i$.*

Since we have the concept of a base space, we also have the notion of a base change.

Definition 1.1.49. *Let X be a scheme, and let $(Y, Y \to X)$ and $(Z, Z \to X)$ (in short, Y and Z) be X-schemes. A **fiber product** or pullback of Y and Z over X, denoted $Y \times_X Z$, to be a scheme, together with morphisms $p_1 : Y \times_X Z \to Y$ and $p_2 : Y \times_X Z \to Z$ that satisfies the following property:*

1. *The diagram*

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{p_2} & Z \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

   is commutative;

2. *For any scheme W, and for any morphisms $f : W \to Y$ and $g : W \to Z$ which makes a commutative diagram*

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \longrightarrow & X, \end{array}$$

   there exists a unique morphism $\theta : W \to Y \times_X Z$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$.

*The morphisms p_1 and p_2 are called the **projection morphisms** of the fiber product onto its factors.*

Theorem 1.1.50. *For any X-schemes Y and Z, the fiber product $Y \times_X Z$ exists, and is unique up to isomorphism.*

Proof. The uniqueness can easily be deduced from the universal property of a pullback in a general category. For existence, the idea is first to construct products for affine schemes and the glue. Let us note that if $(Y \times_X Z)$ exists, ,then for any open subscheme U of Y, the fiber product of U and Y also exists. It suffices for this to take $U \times_X Z := p_1^{-1}(U)$, and the projection morphism can be taken the restrictions of p_1 and p_2 to the open subset $p^{-1}(U)$. furthermore, given the symmetry of the definition, if $(Y \times_X Z, p_1, p_2)$ exists, then $(Z \times_X Y, p_2, p_1)$ is the fiber product of Z and Y.

- Let us first suppose that $X = Spec\ A$, $Y = Spec\ B$ and $Z = Spec\ Z$ are affine schemes. Set $W = Spec(B \otimes_A C)$ and p (resp. q) be the morphism corresponding to the canonical homomorphism $B \to B \otimes_A C$ with $b \mapsto b \otimes 1$ (resp. $C \to B \otimes_A C$ with $c \mapsto 1 \otimes c$). Since the tensor product of rings is pushout and $Spec$ is a contravariant equivalence of categories, (W, p, q) is the fiber product of Y and Z over X.

- Let us now suppose X and Z are affine schemes, and Y arbitrary. Let $\{Y_i\}_{i \in I}$ be a covering of Y by affine open subschemes. From the first case, the fiber product $(Y_i \times_X Z, p_i, q_i)$ exists for every $i \in I$. For any pair i, j, the pre-image $p_i^{-1}(Y_i \cap Y_j)$ and $p_j^{-1}(Y_i \cap Y_j)$ are canonically isomorphic to $(Y_i \cap Y_j) \times_X Z$, which gives an isomorphism of fiber products $f_{ij} : p_i^{-1}(Y_i \cap Y_j) \to p_j^{-1}(Y_i \cap Y_j)$. From the uniqueness of the isomorphism of fiber products $p_i^{-1}(Y_i \cap Y_j \cap Y_k) \times_X Z \cong p_k^{-1}(Y_i \cap Y_j \cap Y_k) \times_X Z$, we have $f_{ik} = f_{jk} \circ f_{ij}$. Then, we can glue the X-schemes $Y_i \times_X Z$ to an X-scheme W by Lemma 1.1.34. As each $Y_i \times_X Z$ can be considered as a Y-scheme and a Z-scheme via the projection morphisms, and as the f_{ij} are compatible with the structures of Y-schemes and of Z-schemes, we obtain projection morphisms $p : W \to X$, $q : W \to Y$ by gluing. Then (W, p, q) is the fiber product of Y and Z over X.

- Let us now suppose X is affine, and Y, Z arbitrary. We cover Z by affine open subscheme Z_i. Then the fiber products of Y and Z_i exist by symmetry of the fiber product. By gluing the $Y \times_X Z_i$ as above, we obtain the existence of the fiber product.

- Let us suppose that X, Y and Z are arbitrary schemes. Let $\{X_i\}_{i \in I}$ be an affine open covering of X. Let $f : Y \to X$ and $g : Z \to X$ be the structural morphisms, $Y_i := f^{-1}(X_i)$, and $Z_i := g^{-1}(X_i)$. Note that any X_i- scheme is an X-schemes in a natural way. It follows, the fiber product of Y_i and Z_i over X_i is also their fiber product over S. Thus, repeats the same argument, the fiber product $(Y \times_X Z, p, q)$ exists.

□

An important application of fiber products is to the notion of base change. Let X be a fixed scheme, and let $Y \to X$ and $Y' \to X$ be morphism of schemes. Then there is a pullback $X' = Y \times_X Y'$ and two projection morphisms $p_1 : Y \times_X Y' \to Y$ and $p_2 : Y \times_X Y' \to Y$. We say that $p_1 : Y \times_X Y' \to Y$ is the **base change** of the morphism $Y' \to X$ via the morphism $Y \to X$.

Definition 1.1.51. *Let $f : Y \to X$ be a morphism of schemes. The* ***diagonal morphism*** *is a morphism $\Delta : Y \to Y \times_X Y$ whose composition with both projection maps $p_1, p_2 : Y \times_X Y \to Y$ is the identity map of Y, i.e., $p_1 \circ \Delta = p_2 \circ \Delta = id_Y$. By universal property of fiber product, the diagonal morphism is unique.*

Definition 1.1.52. *A morphism of schemes $f : Y \to X$ is* ***separated*** *if its diagonal morphism Δ is a closed immersion. In this case, we say X is* ***separated over*** *Y.*

Proposition 1.1.53. *If $f : Y \to X$ is a morphism of affine schemes, then f is separated.*

Proof. Let $Y = Spec\ A$, $X = Spec\ B$. Then A is a B-algebra, and $Y \times_X Y = Spec\ A \otimes_B A$ is also affine. The diagonal morphism Δ is induced by diagonal homomorphism

$$\begin{aligned} A \otimes_B A &\longrightarrow A \\ a \otimes a' &\longmapsto aa'. \end{aligned}$$

This is a surjective homomorphism of rings, since $a \otimes 1 \mapsto a$. Hence Δ is a closed immersion. □

Definition 1.1.54. *A morphism $f : Y \to X$ of schemes is* ***quasi-separated*** *if the diagonal map $\Delta : Y \to Y \times_X Y$ is quasi-compact.*

Quasi-separated morphisms were introduced by Grothendieck and Dieudonné as a generalization of separated morphisms.

Proposition 1.1.55. *A closed immersion is quasi-compact.*

Proof. Let $f : Y \to X$ be a closed immersion, and let $\{V_i\}_{i \in I}$ be an open covering by affine subschemes. Since $f(Y) \cap V_i$ is a closed subset of V_i and V_i is quasi-compact (by Proposition 1.1.28), $f(Y) \cap V_i$ is quasi-compact. Because of $f : Y \to f(Y)$ is a homeomorphism, we have that $f^{-1}(V_i) = f^{-1}(V_i \cap f(Y))$ is quasi-compact as desired. □

Corollary 1.1.56. *A separated morphism of schemes is quasi-separated.*

Definition 1.1.57. *A morphism $f : Y \to X$ of schemes is of* ***finite presentation*** *if f is locally finite presentation, quasi-compact, and quasi-separated.*

Definition 1.1.58.

1. *A morphism $f : Y \to X$ is* ***universally closed*** *if it is closed, and for any morphism $Y' \to X$, the corresponding morphism $f' : Y \times_X Y' \to Y'$ obtained by base change is also closed.*
2. *A morphism $f : Y \to X$ is* ***proper*** *if it is separated, of finite type, and universally closed.*

We summarize some immediate observations of these morphisms.

Remark 1.1.59. *Let $f : Y \to X$ be a morphism of schemes.*

1. *If X and Y are affine schemes, then f is separated, quasi-compact.*
2. *If f is finite, then f is of finite-type.*
3. *If f is of finite presentation at some point $y \in Y$, then f is of finite-type at y.*
4. *If X is locally noetherian, then f is of finite presentation at $y \in Y$ if and only if f is of finite presentation at y.*
5. *If f is separated, then f is quasi-separated.*

We will not present the projective scheme or other fundamental concepts of algebraic geometry, such as normality and singularity.

1.2 Abelian Category

In this section, we introduce the fundamental concepts of abelian categories, with the main references being [4], [35], and [32].

Definition 1.2.1. *A **pre-additive category** is a category $\mathcal{C}$ together with an abelian group structure on each set $hom_{\mathcal{C}}(A, B)$ of morphisms such that the composition maps*

$$\circ : hom_{\mathcal{C}}(A, B) \times hom_{\mathcal{C}}(B, C) \longrightarrow hom_{\mathcal{C}}(A, C)$$
$$(f, g) \longmapsto g \circ f$$

are group homomorphism for all objects $A, B, C \in \mathcal{C}$.

Definition 1.2.2. *Given two objects A, B in a pre-additive category $\mathcal{A}$, **a biproduct** of A and B is a quintuple (P, p_A, p_B, s_A, s_B) such that*

1. *P is an object in $\mathcal{A}$;*
2. *$p_A : P \to A$, $p_B : P \to B$, $s_A : A \to P$ and $s_B : B \to P$ are morphisms in $\mathcal{A}$;*
3. *$p_A \circ s_A = id_A$, $p_B \circ s_B = id_B$, $p_A \circ s_B = 0$, and $p_B \circ s_A = 0$;*
4. *and $s_A \circ p_A + s_B \circ p_B = id_P$.*

The object P is written by $A \oplus B$ in general.

Proposition 1.2.3. *Let $\mathcal{A}$ be a pre-additive category, and let A, B be objects in $\mathcal{A}$. Then, the following statements hold*

1. *If (P, p_A, p_B, s_A, s_B) is a biproduct, then (P, p_A, p_B) is a product of A, B, and (P, s_A, s_B) is a coproduct of A, B.*
2. *The biproduct of A, B exists, if and only if, the product (resp. coproduct) of A, B exists.*

Proof. See Proposition 1.2.4 of [4]. □

Definition 1.2.4. *A pre-additive category is **additive** if it has a zero object and every finite set of objects has a biproduct.*

Definition 1.2.5. *A functor $F : \mathcal{A} \to \mathcal{B}$ between pre-additive categories is **additive** if it is a group homomorphism on each hom-set in $\mathcal{A}$.*

Proposition 1.2.6. *A functor between addtive categories is additive if and only if it preserves all biproduct diagrams.*

Proof. See Proposition 1.3.4 of [4]. □

Definition 1.2.7. *A category is **abelian** if it is additive and*

1. *it has all kernels and cokernels,*
2. *every monomorphism is the kernel of some morphism, and*
3. *every epimorphism is the cokernel of some morphism.*

Theorem 1.2.8. *A category $\mathcal{A}$ is abelian if and only if it is additive, if all kernels and cokernels exist, and if the natural map* $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ *is an isomorphism for all morphisms* $f \in \mathcal{A}$.

This means that the first isomorphism theorem holds in the abelian category.

Proposition 1.2.9. *Let $f : A \to B$ be a morphism in an abelian category, then*

1. *f is a monomorphism if and only if* $\mathrm{Ker}(f) = 0$.
2. *f is an epimorphism if and only if* $\mathrm{Coker}(f) = 0$.

Proof. See Proposition 1.5.4 of [4]. □

Given all these properties, the notion of an exact sequence is well-defined in the abelian category.

Definition 1.2.10. *Let $\mathcal{A}$ be an abelian category. A sequence*

$$... \to A_{-2} \xrightarrow{f_{-1}} A_{-1} \xrightarrow{f_0} A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \to ...$$

*of $\mathcal{A}$ is said to be **exact** if* $\mathrm{Im}\ (f_i) = \mathrm{Ker}\ (f_{i+1})$.

Definition 1.2.11. ***A short exact sequence** is an exact sequence of the form*

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$$

which means that the map f is a monomorphism, g is an epimorphism, and $\mathrm{Ker}\ (g) = \mathrm{Im}\ (f)$.

Definition 1.2.12. *A functor is said to be **left-exact** (resp. **right-exact**) if it preserves all finite limits (resp. colimits).*

This is the general definition of a left (right) exact functor. In the context of abelian categories, the concepts of left and right exactness coincide with the standard definitions, which we will state in the following proposition..

Proposition 1.2.13. *Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{A}'$ be a covariant functor. Then, F is*

1. *left-exact if and only if whenever*

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$$

is exact then

$$0 \longrightarrow F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'')$$

is exact;

2. *right-exact if and only if whenever*

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$$

is exact then

$$F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'') \longrightarrow 0$$

is exact.

Proof. See Proposition 1.11.2 of [4]. □

Definition 1.2.14. *A covariant functor between abelian categories is said to be* ***exact*** *if it is both left exact and right exact.*

The contravariant version is similar.

Proposition 1.2.15. *Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories, and let $G : \mathcal{A} \to \mathcal{A}'$ be a contravariant functor. Then, G is*

1. *left-exact if and only if whenever*

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$$

is exact then

$$0 \longrightarrow G(A'') \xrightarrow{G(g)} G(A) \xrightarrow{G(f)} G(A')$$

is exact;

2. *right-exact if and only if whenever*

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$$

is exact then

$$G(A'') \xrightarrow{G(g)} G(A) \xrightarrow{G(f)} G(A') \longrightarrow 0$$

is exact.

Remark 1.2.16. *Let $F : \mathcal{A} \to \mathcal{A}'$ be a functor between abelian categories. If F is either left or right exact, then it preserves either the biproduct or the bicoproduct. Therefore, by Proposition 1.2.6, F is an additive functor.*

Now we will state the famous snake lemma without any proof.

Proposition 1.2.17. *In an abelian category, If the following diagram is commutative,*

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

where 0 is the zero object. And if the rows are exact sequences, then there is an exact sequence relating the kernels and cokernels of $a,\ b$, and c:

$$\text{Ker } a \to \text{Ker } b \to \text{Ker } c \xrightarrow{d} coker\ a \to coker\ b \to coker\ c$$

*where d is a morphism, called **connecting morphism**.*

Here, we present the cochain complex and its associated cohomology groups in the abelian category.

Definition 1.2.18.

- *A **cochain complex** A^* in an abelian category $\mathcal{A}$ is a sequence*

$$A^* : ... \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to ...$$

of $\mathcal{A}$ such that for each $n \in \mathbb{Z}$, $d^{n+1} \circ d^n = 0$ for all n. So there is a factorization

$$A^n \to \text{Im }(d^n) \to \text{Ker }(d^{n+1}) \to A^{n+1}.$$

- *A **morphism of cochain complexes** $f : A^* \to B^*$ is a family of morphisms $(f^n)_{n\in\mathbb{Z}}$ such that all the diagrams*

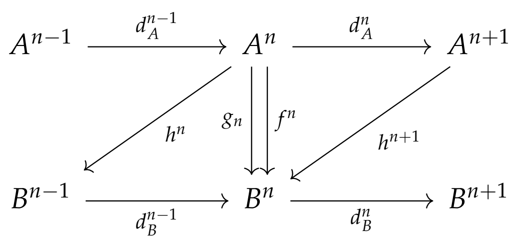
$$\begin{array}{ccc} A^n & \xrightarrow{d_A^n} & A^{n+1} \\ \downarrow f^n & & \downarrow f^{n+1} \\ B^n & \xrightarrow{d_B^n} & B^{n+1} \end{array}$$

are commutative.

- *The* ***category of cochain complexes*** *of an abelian category* $\mathcal{A}$ *is denoted by* $\boldsymbol{CoCh}(\mathcal{A})$.

- *A* ***homotopy*** h *between a pair of morphisms of cochain complexes* $f,\ g : A^* \to B^*$ *is a family of morphisms* $(h^n : A^n \to B^{n-1})$ *such that*

$$f^n - g^n = d_B^{n-1} \circ h^n + h^{n+1} \circ d_A^n$$



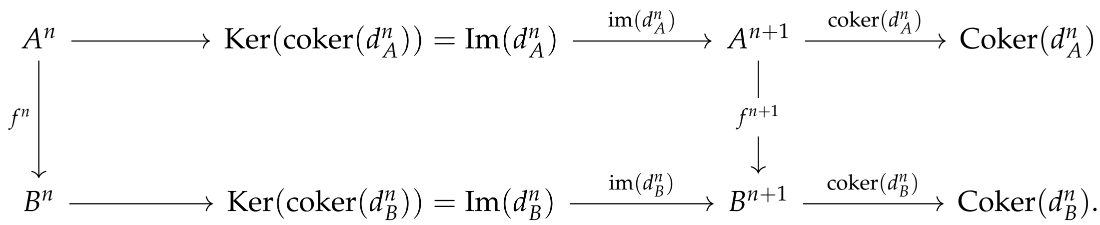
for all n*, this diagram is not necessarily commutative. Two morphisms* $f,\ g : A^* \to B^*$ *are said to be* ***homotopic*** *if a homotopy between* f *and* g *exists, and is denoted by* $f \sim g$.

- *A morphism* $f : A^* \to B^*$ *of cochain complexes is a* ***homotopy equivalence*** *if there exists a morphism* $g : B^* \to A^*$ *such that there exists a homotopy between* $f \circ g$ *and* id_{B^*}*, and there exists a homotopy between* $g \circ f$ *and* id_{A^*}*. If there exists a homotopy equivalence between* A^* *and* B^**, then we say that* A^* *and* B^* *are* ***homotopy equivalent****.*

- *For any* $n \in \mathbb{Z}$*, the n-th* ***cohomology group*** *of a cochain complex* A^* *is the quocient*

$$H^n(A^*) = \frac{\text{Ker }(d^n)}{\text{Im }(d^{n-1})} = \text{Coker}(\text{Im }(d^{n-1}) \to \text{Ker }(d^n)).$$

A morphism $f : A^* \to B^*$ induces functionally, for each n, a morphism $H^{n+1}(f) = f^{*,n+1} : H^{n+1}(A^*) \to H^{n+1}(B^*)$ of $\mathcal{A}$ (or simply f^*) To show this, we see the following commutative diagram



The definition of morphism of cochain complexes and the universal property of $\text{Coker}(d_A^n)$ provide a unique morphism $\mu : \text{Coker}(d_A^n) \to \text{Coker}(d_B^{n+1})$ with $\mu \circ \text{coker}(d_A^n) = \text{coker}(d_B^n) \circ f^{n+1}$. But then $\text{coker}(d_B^n) \circ f^{n+1} \circ \text{im}(d_A^n) = \mu \circ \text{coker}(d_A^n) \circ \text{im}(d_A^n) = 0$, from the universal property of $\text{Ker}(\text{coker}(d_B^n))$, there exists a unique morphism $\nu : \text{Ker}(\text{coker}(d_A^n)) \to \text{Ker}(\text{coker}(d_B^n))$ with

$\mathrm{im}(d_B^n) \circ \nu = f^{n+1} \circ \mathrm{im}(d_A^n)$. By the same argument, since

$$d_B^{n+1} \circ f^{n+1} \circ \ker(d_A^{n+1}) = f^{n+2} \circ d_A^{n+1} \circ \ker(d_A^{n+1}) = 0,$$

there is a unique morphism $\eta : \mathrm{Ker}(d_A^{n+1}) \to \mathrm{Ker}(d_B^{n+1})$ with $\ker(d_B^{n+1}) \circ \eta = f^{n+1} \circ \ker(d_A^{n+1})$.

With all this, we obtain a diagram

$$\begin{array}{ccccccc}
A^n & \longrightarrow & \mathrm{Im}(d_A^n) & \xrightarrow{\psi} & \mathrm{Ker}(d_A^{n+1}) & \xrightarrow{\ker(d_A^{n+1})} & A^{n+1} \\
{\scriptstyle f^n}\downarrow & & {\scriptstyle \nu}\downarrow & & {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle f^{n+1}} \\
B^n & \longrightarrow & \mathrm{Im}(d_B^n) & \xrightarrow{\psi'} & \mathrm{Ker}(d_B^{n+1}) & \xrightarrow{\ker(d_B^{n+1})} & B^{n+1}.
\end{array}$$

We will prove that this diagram commutes, i.e., $\eta \circ \psi = \psi' \circ \nu$. Note that

$$\ker(d_B^{n+1}) \circ \psi' \circ \nu = \mathrm{im}(d_B^n) \circ \nu = f^{n+1} \circ \mathrm{im}(d_A^n) = f^{n+1} \circ \ker(d_A^{n+1}) \circ \psi = \ker(d_B^{n+1}) \circ \eta \circ \psi.$$

Since $\ker(d_B^{n+1})$ is a monomorphism, we have $\psi' \circ \nu = \eta \circ \psi$. So, from the universal property of the $\mathrm{Coker}(\psi)$, there is a unique $f^{*,n+1} : H^{n+1}(A^*) = \mathrm{Coker}(\psi) \to H^{n+1}(B^*) = \mathrm{Coker}(\psi')$ making the diagram

$$\begin{array}{ccccc}
\mathrm{Im}(d_A^n) & \xrightarrow{\psi} & \mathrm{Ker}(d_A^{n+1}) & \xrightarrow{\mathrm{coker}(\psi)} & \mathrm{Coker}(\psi) \\
{\scriptstyle \nu}\downarrow & & {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathrm{Im}(d_B^n) & \xrightarrow{\psi'} & \mathrm{Ker}(d_B^{n+1}) & \xrightarrow{\mathrm{coker}(\psi')} & \mathrm{Coker}(\psi')
\end{array}$$

commute.

By the uniqueness, we have $(f \circ g)^{*,n} = f^{*,n} \circ g^{*,n}$, $id^{*,n} = (id_{H^n(A^*)})$, and $(f+g)^{*,n} = f^{*,n} + g^{*,n}$. So $H^n : \mathbf{CoCh}(\mathcal{A}) \to \mathcal{A}$ is an additive covariant functor.

Proposition 1.2.19. *If two morphisms $f,\ g : A^* \to B^*$ are homotopic, then*

$$f^{*,n} = g^{*,n} : H^n(A^*) \to H^n(B^*).$$

Proof. We will prove that $(f-g)^{*,n} = 0$ for each n. From the previous construction, $(f-g)^{*,n}$ is induced by a commutative diagram

$$\begin{array}{ccccccc} A^{n-1} & \longrightarrow & \mathrm{Im}(d_A^{n-1}) & \xrightarrow{\psi} & \mathrm{Ker}(d_A^n) & \xrightarrow{\ker(d_A^n)} & A^n \\ \Big\downarrow{\scriptstyle (f-g)^{n-1}} & & \Big\downarrow{\scriptstyle \nu} & & \Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle (f-g)^n} \\ B^{n-1} & \longrightarrow & \mathrm{Im}(d_B^{n-1}) & \xrightarrow{\psi'} & \mathrm{Ker}(d_B^n) & \xrightarrow{\ker(d_B^n)} & B_n. \end{array}$$

Let h be a homotopy between f and g, by the definition, we have

$$\begin{aligned} \pi := & (f-g)^n \circ \ker(d_A^n) = (h^{n+1} \circ d_A^n + d_B^{n-1} \circ h^n) \circ \ker(d_A^n) \\ & = d_B^{n-1} \circ h^n \circ \ker(d_A^n) = \ker(d_B^n) \circ \psi' \circ (\mathrm{coim}(d_B^{n-1}) \circ h^n \circ \ker(d_A^n)). \end{aligned}$$

Let $\pi' := \mathrm{coim}(d_B^{n-1}) \circ h^n \circ \ker(d_A^n)$, so that $\pi = \ker(d_B^n) \circ \psi' \circ \pi'$. On the other hand, $\pi = \ker(d_B^n) \circ \eta$, since $\ker(d_B^n)$ is a monomorphism, we have $\eta = \psi' \circ \pi'$. Then we obtain

$$(f-g)^{*,n} \circ \mathrm{coker}(\psi) = \mathrm{coker}(\psi') \circ \eta = \mathrm{coker}(\psi') \circ \psi' \circ \pi' = 0.$$

Since $\mathrm{coker}(\psi)$ is an epimorphism, we conclude that $(f-g)^* = 0$. □

Corollary 1.2.20. *If $f : A^* \to B^*$ is a homotopy equivalence, then $f^{*,n}$ is an isomorphism for every $n \in \mathbb{Z}$.*

Proof. From the definition of homotopy equivalence, there is a morphism $g : B^* \to A^*$ of cochain complexes such that $f \circ g \sim id_{B^*}$ and $g \circ f \sim id_{A^*}$. By the previous proposition, we have $f^{*,n} \circ g^{*,n} = (f \circ g)^{*,n} = (id_{H^n(B^*)})$ and $g^{*,n} \circ f^{*,n} = (g \circ f)^{*,n} = (id_{H^n(A^*)})$ as desired. □

Proposition 1.2.21. *1. The category **CoCh**$(\mathcal{A})$ is an abelian category.*

2. A sequence of cochain complexes

$$0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$$

is exact if and only if

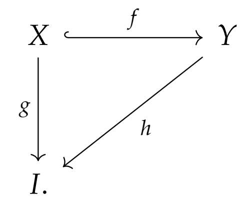
$$0 \to A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \to 0$$

is exact for every $n \in \mathbb{Z}$.

Proof. For item 1), see 1.2.3 of [35]. The item 2) follows from the definition of the morphism of cochain complexes. □

Now, we will introduce the right exact functor and discuss some basic facts about it.

Definition 1.2.22. *An object I in a category C is said to be* ***injective*** *if for every monomorphism $g : X \to Y$ and every morphism $g : X \to I$ there exists a morphism $h : Y \to I$ (no need to be unique) extending g to Y, i.e., the following diagram commutes*



Proposition 1.2.23. *Let $\mathcal{A}$ be an abelian category. If*

$$0 \to I \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence in $\mathcal{A}$ such that I is injective, then the sequence splits, i.e., there is an isomorphism h from B to the direct sum of I and C, such that $h \circ f$ is the natural injection of A into the direct sum, and $g \circ h^{-1}$ is the natural projection of the direct sum onto C, so the sequence

$$0 \to I \xrightarrow{h \circ f} I \oplus C \xrightarrow{g \circ h^{-1}} C \to 0$$

is exact.

Proof. From the properties of a short exact sequence, the morphism f is a monomorphism. By the definition of an injective object, there is a $r : B \to I$ such that $r \circ f = id_I$. Now consider morphism $id_B - f \circ r : B \to B$. We have

$$(id_B - f \circ r) \circ f = f - f \circ r \circ f = f - f = 0,$$

which implies that there exists a unique morphism $s : B \to C$ such that $s \circ g = id_B - f \circ r$, since $g = \text{coker}(f)$. We have already $f \circ r + s \circ g = id_B$, $g \circ f = 0$, and $r \circ f = id_I$. We have also

$$g \circ s \circ g = g \circ (id_B - f \circ r) = g - g \circ f \circ r = g = id_C \circ g,$$

hence $g \circ s = id_C$, since g is an epimorphism. Finally

$$r \circ s \circ g = r \circ (id_B - f \circ r) = r - r \circ f \circ r = r - r = 0,$$

thus $r \circ s = 0$, since g is an epimorphism. This concludes that the quintuple (B, r, g, f, s) is the biproduct of I and C. □

Definition 1.2.24. *An abelian category $\mathcal{A}$ is said to have **sufficiently many injective objects** if for every object $X \in \mathcal{A}$ there exists a monomorphism from A into an injective object in $\mathcal{A}$.*

Example 1.2.25. *The category **Ab** of abelian groups is an abelian category which has sufficiently many injective objects.*

Definition 1.2.26. *Let $\mathcal{A}$, $\mathcal{A}'$ be two abelian categories. A covariant ∂-**functor** from $\mathcal{A}$ to $\mathcal{A}'$ is a system $F = (F^i)_{i \geq 0}$ of covariant exact functors*

$$F^i : \mathcal{A} \to \mathcal{A}'$$

together with a connecting morphism $\partial : F^i(A'') \to F^{i+1}(A')$ defined for each $i \geq 0$ and each short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, satisfying the following properties:

1. *For every commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

   in $\mathcal{A}$, the diagram

$$\begin{array}{ccc} F^i(A'') & \xrightarrow{\partial} & F^{i+1}(A') \\ \downarrow & & \downarrow \\ F^i(B'') & \xrightarrow{\partial} & F^{i+1}(B') \end{array}$$

   is commutative for all $i \geq 0$.

2. *For every exact sequence*

$$0 \to A' \to A \to A'' \to 0$$

   in $\mathcal{A}$ the long sequence

$$0 \to F^0(A') \to F^0(A \to F^0(A'') \to F^1(A') \to F^1(A) \to ...$$

   is exact in $\mathcal{A}'$.

Definition 1.2.27. *Let $F = (F^i)_{i \geq 0}$ and $F' = (F^i)_{i \geq 0}$ be two ∂-functors from an abelian category $\mathcal{A}$ to* *another abelian category $\mathcal{A}'$.* ***A morphism from** F **to** F′ is a system $f = (f^i)_{i \geq 0}$ of functorial morphisms*

$$f^i : F^i \to (F')^i$$

which satisfies the following property:
For any exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$ the diagram

$$\begin{array}{ccc} F^i(A'') & \longrightarrow & F^{i+1}(A') \\ \Big\downarrow{\scriptstyle f^i(A'')} & & \Big\downarrow{\scriptstyle f^{i+1}(A')} \\ (F')^i(A'') & \longrightarrow & (F')^{i+1}(A') \end{array}$$

commutes.

Definition 1.2.28. *A ∂-functor $F = (F^i)_{i \geq 0}$ from the abelian category $\mathcal{A}$ to the abelian category $\mathcal{A}'$ is called **universal** if each morphism $f^0 : F^0 \to (F')^0$ of functors has a unique extension to a morphism $f : F \to F'$ of ∂-functors.*

Definition 1.2.29. *An additive covariant functor $F : \mathcal{A} \to \mathcal{A}'$ from an abelian category to an additive category is called **effaceable** if for every object $A \in \mathcal{A}$, there is a monomorphism $m : A \to M$ in $\mathcal{A}$ such that $F(m) = 0$*

Theorem 1.2.30. *Let $\mathcal{A}$ be an abelian category with sufficiently many injective objects, then*

1. *A functor $F : \mathcal{A} \to \mathcal{A}'$ is effaceable if and only if $F(M) = 0$ for all injective objects $M \in \mathcal{A}$;*

2. *An exact ∂-functor $F = (F^i)$ from $\mathcal{A}'$ to an abelian category $\mathcal{A}'$ is universal if and only if F^i is effaceable for every $i > 0$.*

Proof. (1) : Assume that F is effaceable, and let M be an injective object in $\mathcal{A}$. By definition, there is a monomorphism $m : M \to N$ in $\mathcal{A}$ with $F(m) = 0$. From the definition of the injective object, there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\;m\;} & N \\ \Big\downarrow{\scriptstyle id_M} & \swarrow{\scriptstyle n} & \\ M. & & \end{array}$$

This implies $id_{F(M)} = F(id_M) = F(n) \circ F(m) = 0$, so $F(M) = 0$. The converse is obvious, since $\mathcal{A}$ has sufficient many injective, for every object $A \in \mathcal{A}$, there is a monomorphism $m : A \to M$ with M injective. Since $F(M) = 0$, $F(m) = 0$.

(2) : see 2.1.2 of [32]. □

Definition 1.2.31. *Let $F : \mathcal{A} \to \mathcal{A}'$ be a left exact additive covariant functor between abelian categories. The* ***right derived functor of*** *F is the universal exact ∂-functor from $\mathcal{A}$ to $\mathcal{A}'$ that extends F. If it exists, it is unique up to isomorphism. We will denote it by $(R^iF)_{i\geq 0}$, and R^iF is called the i-th right derived functor of F.*

Theorem 1.2.32. *Let $\mathcal{A}$ be an abelian category with sufficiently many injective objects, and let $\mathcal{A}'$ be an abelian category. Then for every left exact additive covariant functor $F : \mathcal{A} \to \mathcal{A}'$ the right derived functor $(R^iF)_{i\geq 0}$ exists.*

Proof. Since $\mathcal{A}$ has sufficiently many injective objects, every object $A \in \mathcal{A}$ has an **injective resolution**, i.e., there is an exact sequence

$$M^*(A) : 0 \to A \to M^0 \to M^1 \to ...$$

where M^i are injective objects in $\mathcal{A}$.

We will use the following facts from [6], ch. V:

1. If $M^*(A)$ and $M^*(A')$ are injective resolutions of A and A' in $\mathcal{A}$, then every morphism $u : A \to A'$ extends to a morphism $M^*(A) \to M^*(A')$ of $\mathbf{CoCh}(\mathcal{A})$, and any two extensions of u are homotopic. In particular, the injective resolution $M^*(A)$ of A is uniquely determined up to homotopy equivalent.

2. Any exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ extends to an exact sequence

$$0 \to M^*(A') \to M^*(A) \to M^*(A'') \to 0$$

   in $\mathbf{CoCh}(\mathcal{A})$.

Given an object $A \in \mathcal{A}$, and an injective resolution of A

$$M^*(A) : 0 \to A \to M^0 \to M^1 \to ....$$

Since F is left exact which preserves kernel, the sequence

$$F(M^*(A)) : F(0) \to F(A) \to F(M^0) \to F(M^1) \to ...$$

is a cochain complexes, and since additive functor preserves homotopy, $F(M^*(A))$ is uniquely determined up to the homotopy equivalence, i.e., for any injective resolutions $M^*(A)$ and $M^*(A)'$

of A, the cochain complexes $F(M^*(A))$ and $F(M^*(A)')$ are homotopy equivalent. Hence, the system of functor given by

$$R^0F(A) = H^0(F(M^*(A))) = \text{Ker } (F(M^0) \to F(M^1))$$
$$R^iF(A) = H^i(F(M^*(A))) = \frac{\text{Ker } (F(M^i) \to F(M^{i+1}))}{\text{Im } (F(M^{i-1}) \to F(M^i))}, \quad i \geq 1$$

is well-defined. Moreover, for any morphism $u : A \to A'$ in $\mathcal{A}$, we have a unique extension $M^*(u) : M^*(A) \to M^*(A')$ up to homotopy, so the morphism $R^iF(u) = H^i(M^*(u)) : R^iF(A) \to R^iF(A')$ is also well-defined. For this reason, R^iF is an additive covariant functor.

Since F is left exact, $R^0F = F$. For each $i > 0$ the R^iF are effaceable, since for an injective object $M \in \mathcal{A}$ an injective resolution of M is given by $0 \to M \xrightarrow{id_M} M \to 0$, from which we have $R^iF(M) = 0$ for each $i > 0$, so by Theorem 1.2.30, if $(R^iF)_{i \geq 0}$ is a ∂-functor, then it is universal.

Now, we will prove that R^iF is a ∂-functor. Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0,$$

from 2), it is extendable to a short exact sequence

$$0 \to M^*(A') \to M^*(A) \to M^*(A'') \to 0$$

for suitably chosen resolutions. Since $M^i(A')$ is injective, all exact sequences

$$0 \to M^i(A') \to M^i(A) \to M^i(A'') \to 0$$

split and therefore

$$0 \to F(M^i(A')) \to F(M^i(A)) \to F(M^i(A'')) \to 0$$

is exact. The exact sequence

$$0 \to F(M^*(A')) \to F(M^*(A)) \to F(M^*(A'')) \to 0$$

of complexes in $\mathcal{A}'$ yields. The connecting homomorphisms

$$\partial : R^iF(A'') \to R^{i+1}F(A')$$

are provided by the snake lemma, so that the long cohomology sequence becomes exact and the ∂'s functorial for short exact sequences in $\mathcal{A}$. Hence, $(R^iF)_{i \geq 0}$ is a right derived functor of F. □

From the construction, we can interpret that the right derived functors $(R^iF)_{i\geq 0}$ "measure" the failure of exactness of the left-exact functor F.

Definition 1.2.33. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories having sufficiently many injectives, and let $F : \mathcal{A} \to \mathcal{B}$ be an additive covariant functor. An F-**acyclic object**, is an object X in $\mathcal{A}$ such that*

$$R^rF(X) = 0 \quad \textit{for all } i > 0.$$

Remark 1.2.34. *In the proof of the previous theorem,, we saw that for any left exact additive functor F, every injective object is an F-acyclic object.*

Lemma 1.2.35. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $G : \mathcal{A} \to \mathcal{B}$ be a right adjoint additive functor with $F : \mathcal{B} \to \mathcal{A}$ left adjoint to G. Then the following conditions are equivalent:*

1. *The functor F preserves injective maps.*
2. *The functor F is exact.*
3. *The functor G preserves injectives, i.e., sends injective object into injective object.*

Proof. See Lemma 12.29.1 of [31]. □

Proposition 1.2.36. *Let L_1 and L_2 be left exact functors from abelian categories with enough injective objects. And let $L = L_2 \circ L_1$. If L_1 preserves injective objects and X is an L_1-acyclic object, then*

$$(R^rL)(X) = (R^rL_2)(L_1X).$$

In particular, the above equality holds if L_1 is an exact functor that preserves injective objects.

Proof. Let

$$0 \to X \to I_0 \to I_1 \to ...$$

be an injective resolution of X. By hypotheses on L_1,

$$0 \to L_1X \to L_1I_0 \to L_1I_1 \to ...$$

is an injective resolution of L_1X, which can be used to compute the $(R^rL_2)(L_1X)$. Now we have

$$R^rL(X) = \frac{\text{Ker } (L(I^r) \to L(I^{r+1}))}{\text{Im } (L(I^{r-1}) \to L(I^r))} = \frac{\text{Ker } (L_2(L_1I^r) \to L_2(L_1I^{r+1}))}{\text{Im } (L_2(L_1I^{r-1}) \to L_2(L_1I^r))} = R^rL_2(L_1(X))$$

as desired. □

1.3 Grothendieck Topology

In this section, we introduce the Grothendieck topology, sheaves on Grothendieck topology, and sheafification, with the main references being [5], [16], [32], [20], [12], [33], and [10].

To define a sheaf, it is not necessary to have a topological space. Grothendieck shows that it suffices to have a category $\mathcal{C}$ together with, for each object $U \in \mathcal{C}$, a set of families of maps $\{U_i \to U\}_{i\in I}$, called the coverings of U, satisfying the following axioms:

1. For every covering $\{U_i \to U\}_{i\in I}$ and every morphism $V \to U$ in $\mathcal{C}$, the pullbacks (fiber products) $U_i \times_U V$ exist, and $\{U_i \times_U V \to V\}$ is a covering of V;

2. If $\{U_i \to U\}_{i\in I}$ is a covering of U, and for each $i \in I$, $\{V_{ij} \to U_I\}_{j\in J_i}$ is a covering of U_i, then the family $\{V_{ij} \to U_i \to U\}_{i,j}$ is a covering of U;

3. For every object $U \in \mathcal{C}$, the single family $\{id : U \to U\}$ is a covering of U.

These axioms generalize the notion of an open covering in a topological space. For any open subset U and V with $V \subseteq U$, and for any open covering $U = \bigcup_{i\in I} U_i$ of U, the first axiom asserts that the family $\{U_i \cap V : i \in I\}$ forms an open covering of V. The second and third axioms are more straightforward.

Definition 1.3.1. *The system of coverings satisfying the above axioms is called a* ***Grothendieck pre-topology****, or simply* ***topology****, and $\mathcal{C}$ together with a topology τ is called a* ***site****.*

Definition 1.3.2. *A* ***morphism*** $f : (\mathcal{C}', \tau') \to (\mathcal{C}, \tau)$ ***of topologies*** *is a functor $f : \mathcal{C} \to \mathcal{C}'$ of the underlying categories with the following two properties:*

1. *If $\{U_i \xrightarrow{\phi_i} U\}$ is a covering of $(\mathcal{C}, \tau)$, then $\{f(U_i) \xrightarrow{f(\phi_i)} f(U)\}$ is also a covering of $(\mathcal{C}', \tau')$.*

2. *For each covering $\{U_i \to U\}$ of $(\mathcal{C}, \tau)$ and a morphism $V \to U$ in $\mathcal{C}$, the canonical morphism*

$$f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)$$

   is an isomorphism for all i.

A morphism of topologies $f : (\mathcal{C}', \tau') \to (\mathcal{C}, \tau)$ is actually a functor in the opposite direction, which aligns better with our intuition from topological spaces. Some authors may define a morphism of topologies $f : (\mathcal{C}', \tau') \to (\mathcal{C}, \tau)$ as a functor $\mathcal{C}' \to \mathcal{C}$, as in [32], because of this is the definition used in SGA4 (the seminar, not the book). However, since this approach was not adopted in the published version, we will use the definition given above.

Definition 1.3.3. *A **presheaf of sets on a site** $\mathcal{C}_\tau = (\mathcal{C}, \tau)$ is a contravariant functor $F : \mathcal{C} \to$ **Set**. A **morphism between presheaves** is just a natural transformation, We will denote the category of presheaf of sets on a site $\mathcal{C}_\tau$ by $Psh(\mathcal{C}_\tau)$.*

Definition 1.3.4. *A presheaf F on a site $\mathcal{C}_\tau$ is **separated** if, for any coverings $\{U_i \to U\}$, the canonical map*

$$F(U) \longrightarrow \prod_i F(U_i)$$

is injective.

Notation 1.3.5. *If $\phi : U \to V$ is a morphism of $\mathcal{C}$, then we sometimes denote $F(\phi) : F(V) \to F(U)$ by $a \mapsto a|_U$ (this can be confuse, since there may be more than one morphism from U to V).*

Similarly, a presheaf of abelian groups or rings on $\mathcal{C}_\tau$ is a contravariant functor from $\mathcal{C}$ to the category of abelian group or rings.

The concept of a presheaf on a site does not depend on the coverings, whereas the notion of a sheaf does depend on them.

Definition 1.3.6. *A **sheaf on site** $\mathcal{C}_\tau$ is a presheaf F that satisfies the sheaf condition: the diagram*

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \overset{F(pr_1)}{\underset{F(pr_2)}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(U_i \times_U U_j)$$

is exact (i.e., an equalizer) for every covering $\{U_i \to U\}$, where $(U_i \times_U U_j)$ is the fiber product of $(U_i \to U, U_j \to U)$, and the left morphism is induced by the product.

Hence F is a sheaf if the map

$$F(U) \to \prod F(U_i)$$
$$f \mapsto (f|_{U_i})_{i\in I}$$

identifies $F(U)$ with the subset of the product consisting of families (f_i) such that

$$f_i|_{U_i\times_U U_j} = f_j|_{U_i\times_U U_j}$$

for every $i, j \in I \times I$. We denote the category of sheaves of sets on $\mathcal{C}_\tau$ by $Sh(\mathcal{C}_\tau)$, and the category of sheaves of abelian groups by $Ab(\mathcal{C}_\tau)$. It is easy to see that $Sh(\mathcal{C}_\tau)$ is a full faithful subcategory of $Psh(\mathcal{C}_\tau)$.

Theorem 1.3.7. *The category $Ab(\mathcal{C}_\tau)$ is an abelian category with sufficiently many injective objects.*

Proof. (see [32] Section 3 of Chapter I). □

Hence, the right derived functors $R^q f$ exist for each given left exact additive functor from $Ab(\mathcal{C}_\tau)$ to Ab. For a fixed object $U \in \mathcal{C}$, we consider the section functor $\Gamma_U : Ab(\mathcal{C}_\tau) \to Ab$ defined by $\Gamma_U(F) = F(U)$, this functor is left exact functor, so there is a right derived functor.

Definition 1.3.8. *For an abelian sheaf F on $Ab(\mathcal{C}_\tau)$ we define* ***the q-th (sheaf) cohomology group of U with values in*** *F by*

$$H^q_\tau(U, F) = R^q\Gamma_U(F).$$

The sheaf cohomology "measures" the lack of exactness of the global section functor $\Gamma(X, -)$. In addition, it also "measures" the capability to "lift local data to global" in certain situations. Here's a summary of how it works:

Let X be a topological space and F a sheaf on X. Consider a cover $\{U_i\}_{i\in I}$ of X, and sections $f_i \in F(U_i)$ over each open set U_i. Let $G \subseteq F$ be a subsheaf, and assume that for all $i, j \in I$, the difference $f_i - f_j \in G(U_i \cap U_j)$. One might ask wether f's "glue" to a global section $f \in F(X)$ such that $f|_{U_i} - f_i \in G(U_i)$ for each i. This is a generalization of the Cousin's problem: Given an open cover $\{U_i\}_{i\in I}$ of $\mathbb{C}$, and meromorphic functions f_i defined on each U_i, where $f_i - f_j$ is holomorphic on $U_i \cap U_j$. Is there a meromorphic function f on $\mathbb{C}$ such that $f|_{U_i} - f_i$ is holomorphic for each i?

This problem is closely related to the exactness of the global section functor $\Gamma(X, -)$. Let $\overline{f}_i \in (F/G)(U_i)$ be the image of f_i under the projection $F \to F/G$. The sections $\overline{f}_i$ glue in the quotient sheaf F/G, since $\overline{f}_i - \overline{f}_j = 0 \in (F/G)(U_i \cap U_j)$, and by the sheaf property, there exists an element $\overline{f} \in (F/G)(X)$. Therefore, the f_i's can be lifted to a global section $f \in F(X)$ such that $f|_{U_i} - f_i \in G(U_i)$ if and only if the global section $\overline{f}$ lies in the image of the map $(\Gamma(X, F) \to \Gamma(X, F/G))$.

Since $\Gamma(X, -)$ is left exact, we have the following exact sequence:

$$0 \to \Gamma(X, G) \to \Gamma(X, F) \to \Gamma(X, F/G) \to H^1(X, G).$$

Hence, the f_i's can be lifted to a global section $f \in F(X)$ if and only if the first cohomology group $H^1(X, G)$ is trivial.

There is another approach to sheaves, defined using a Grothendieck topology (rather than just a pretopology). We will explain how this construction works.

Let $\mathcal{C}$ be a category and c be an object in $\mathcal{C}$. A **sieve** on c is a subfunctor of the functor $h_c = hom(-, c)$, which assigns to each object $x \in \mathcal{C}$ the set of morphism from x to c. Let S be a sieve on c, and let $f : c' \to c$ be a morphism in $\mathcal{C}$, We denote the pullback $S \times_{h_c} h_{c'}$ of S along f by f^*S.

More concretely, for each object $x \in \mathcal{C}$, we have

$$f^*S(x) = \{g : x \to c' : f \circ g \in S(x)\}.$$

Definition 1.3.9. *A **Grothendieck topology** J on a category $\mathcal{C}$ is functor from $\mathcal{C}$ to **Set** such that $J(c)$ is a set of distinguished sieves on c (whose element is called a **covering sieve**) for each object $c \in \mathcal{C}$, satisfying the following axioms:*

1. *If $S \in J(c)$ and $f : c' \to c$ is a morphism of $\mathcal{C}$, then the pullback $f^*S \in J(c')$;*
2. *Let $S \in J(c)$,, and let T be any sieve on c. Suppose that for every object $c' \in \mathcal{C}$ and every morphism $f : c' \to c$ in $S(c')$, the pullback sieve $f^*T \in J(c')$. Then $T \in J(c)$;*
3. *For every object $c \in \mathcal{C}$, $h_c \in J(c)$.*

If $\mathcal{C}$ is a category and J is a Grothendieck topology on $\mathcal{C}$, then the pair $(\mathcal{C}, J)$ is also called a site.

These axioms are analogous to the axioms of a Grothendieck pretopology.

For any pretopology, the collection of all sieves that contain some covering family from the pretopology is a Grothendieck topology. So, there is no ambiguity in site.

Definition 1.3.10. *In the sense of Grothendieck topology, a sheaf F on site $(\mathcal{C}, J)$ is a presheaf on $\mathcal{C}$ such that for every object $c \in \mathcal{C}$ and every covering sieve $S \in J(c)$, the natural map $hom(h_c, F) \to hom(S, F)$ induced by the inclusion $S \hookrightarrow h_c$ is a bijection (This definition is equivalent to the previous one).*

There is a consequence of Yoneda's lemma which characterizes the set $hom(R, F)$ even if F is not a sheaf.

Proposition 1.3.11. *Let $\mathcal{C}$ be a category, and let $F : \mathcal{C} \to$ **Set** be a presheaf. If R is a sieve on $X \in \mathcal{C}$, then*

$$hom(R, F) \cong \varprojlim_{(U \to X) \in \mathcal{C}/R} F(U)$$

where $\mathcal{C}/R$ is the comma category.

Proof. See Proposition 4.6 of [23]. □

Given a presheaf $\mathcal{F}$, we can construct a sheaf from it through a process called sheafification. This process defines a functor, and in fact, sheafification is a left adjoint to the inclusion functor from $Sh(\mathcal{C})$ to $Psh(\mathcal{C})$. Let's explore this concept further, particularly in the context of Grothendieck's pre-topology.

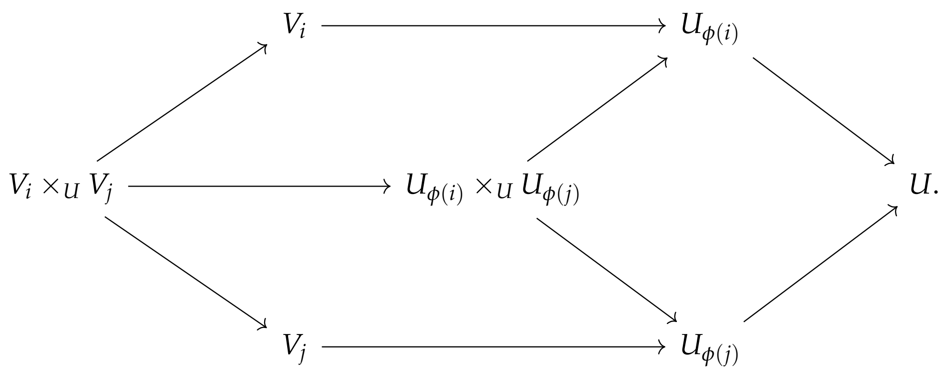
We will work with the site $(\mathcal{C}, \tau)$. For simplicity, we will often denote the site by $\mathcal{C}$ instead of $(\mathcal{C}, \tau)$.

Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_k \to U\}_{k \in I}$ be two coverings of U. We say that the pair $(\phi : K \to I, \{V_k \to U_{\phi(k)}\}_{k \in K})$ is a **refinement** of $\mathcal{U}$ by $\mathcal{V}$, if the compositions $V_k \to U_{\phi(k)} \to U$ is equal to the $V_k \to U$.

Let F be a presheaf of sets on $\mathcal{C}$, and let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of U. We denote the equalizer

$$\left\{ (s_i)_{i \in I} \in \prod_{i \in I} F(U_i) : s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \ \ \forall i, j \in I \right\}$$

by $F(\mathcal{U})$. It is clear that the refinement of $\mathcal{U}$ by $\mathcal{V}$ induces a commutative diagram



For this reason, there is a map $F(\mathcal{U}) \to F(\mathcal{V})$ defined by $(s_i) \mapsto (s_{\phi(k)}|_{V_k})$.

Let I_U be the category of all coverings of U: the objects are the coverings of U in $\mathcal{C}$, and a morphism from $\mathcal{V}$ to $\mathcal{U}$ is the refinement of $\mathcal{U}$ by $\mathcal{V}$. Note that I_U is not empty since $\{id_U\}$ is a covering of U. According to the previous remarks, the construction $\mathcal{U} \mapsto F(\mathcal{U})$ defines a contravariant functor from I_U to **Set**. We define

$$F^+(U) = \varinjlim_{\mathcal{U} \in I_U^{op}} F(\mathcal{U}).$$

Now, we turn the collection of sets $F^+(U)$ into a presheaf. Let $V \to U$ be a morphism in $\mathcal{C}$. From the definition of the covering, there is a natural morphism $I_U \to I_V$ given by

$$\{U_i \to U\} \mapsto \{U_i \times_U V \to V\}.$$

Similarly, there is a functorial map of sets $F(\{U_i \to U\}) \to F(\{U_i \times_U V \to V\})$ defined by $(s_i) \mapsto (s_i|_{U_i \times_U V})$. Hence, by generalities of colimits we obtain a canonical map $F^+(U) \to F^+(V)$. This construction defines a presheaf F^+. Of course, if F is a sheaf on $\mathcal{C}$, then $F^+ \cong F$.

Proposition 1.3.12. *1. If F is a presheaf on $\mathcal{C}$, then F^+ is a separated sheaf.*

2. *If F is a separated sheaf, then F^+ is a sheaf.*

Hence, the presheaf $F^\# := F^{++}$ is always a sheaf.

Proof. see [16] 5.III □

$F^\#$ is called the sheaf associated to F. As noted above, this process defines a functor known as **sheafification**.

$$a_\tau : Psh(\mathcal{C}) \longrightarrow Sh(\mathcal{C})$$
$$F \longmapsto F^\#.$$

Moreover:

Theorem 1.3.13. *The functor a_τ is a left adjoint of the inclusion functor $i : Sh(\mathcal{C}) \to Psh(\mathcal{C})$.*

Proof. see [16] 6.III . □

Corollary 1.3.14. *For any presheaf F on a site T, the following are equivalent:*

1. *F is a sheaf.*
2. *For any covering $\{U_i \to U\}_{i\in I}$ in T there exists a refinement $\{U'_j \to U\}$ of $\{U_i \to U\}$ in T, such that*
$$F(U) \to \prod F(U'_j) \rightrightarrows F(U'_{j_0} \times_U U'_{j_1})$$
is exact.

Proof. See I.3.1.4 of [32] □

Sheafification of presheaves of abelian groups, groups, rings, etc., is defined in the same way, and sheafification is exact when the category is abelian.

Next, we will summarize the canonical topology of a given category.

Definition 1.3.15. *Let $\mathcal{C}$ be a category. Recall, a morphism in $\mathcal{C}$ is called an epimorphism if the canonical map $hom(V,Z) \to hom(U,Z)$ is injective for every object $Z \in \mathcal{C}$. A morphism $U \to V$ is an* ***effective epimorphism****, if the diagram*

$$hom(V,Z) \to hom(U,Z) \rightrightarrows hom(U \times_V U, Z)$$

is exact for every $Z \in \mathcal{C}$, where the two right-hand maps are induced by the projections of $U \times_V U$ onto the left and right factor. A morphism $U \to V$ is called a ***universal effective epimorphism****, if $U \times_V W \to W$ is an effective epimorphism for every morphism $W \to V$ in $\mathcal{C}$.*

These notions generalize to families of morphisms $\{U_i \to V\}_{i\in I}$: A family $\{U_i \to V\}_{i\in I}$ is a ***family of epimorphisms*** *if $\{hom(V,Z) \to \prod_{i\in I} hom(U_i, Z)\}$ is injective for every $Z \in \mathcal{C}$. It is a* ***family of effective epimorphisms*** *if the diagram*

$$hom(V,Z) \to \prod_{i\in I} hom(U_i, Z) \rightrightarrows \prod_{i,j\in I} hom(U_i \times_V U_j, Z)$$

is exact for every object $Z \in \mathcal{C}$. A family of effective epimorphisms is a ***family of universal effective epimorphisms*** *if $\{U_i \times_V W \to W\}$ is a family of effective epimorphisms for every morphism $W \to V$ in $\mathcal{C}$.*

Definition 1.3.16. *Let $\mathcal{C}$ be a category. The* ***canonical topology*** *τ of $\mathcal{C}$ is the collection of all family of universal effective epimorphisms in $\mathcal{C}$.*

Remark 1.3.17. *To show that canonical topology is Grothendieck's (pre-)topology, see I. 1.3 of [32] and IV, 1 of [11]*

There are some immediate facts

Proposition 1.3.18.

1. *Every representable presheaf of sets is a sheaf on $(\mathcal{C}, \tau)$.*
2. *The canonical topology is the finest topology on $\mathcal{C}$ such that all representable presheaves of sets are sheaves.*

Definition 1.3.19. *A* ***Grothendieck topos*** *is a category equivalent to the category of sheaves on a site.*

In this thesis, we will not consider elementary toposes. Thus, we will sometimes use the term "topos" in place of "Grothendieck topos."

The notions of direct image and inverse image functors in the category of sheaves on a topological space generalize to the concept of a geometric morphism in a topos.

Definition 1.3.20. *If E and F are toposes, a* ***geometric morphism*** *$f : E \to F$ consists of a pair of adjoint functors (f^*, f_*)*

$$f^* : F \to E \quad \text{and} \quad f_* : E \to F$$

such that the left adjoint functor f^* preserves finite limits. The left adjoint f^* is called the ***inverse image*** *of the geometric morphism, and the right adjoint f_* is called the* ***direct image*** *of the geometric morphism.*

Theorem 1.3.21. *Let $f : \mathcal{C}'_{\tau'} \to \mathcal{C}_\tau$ be a morphism of sites, with $\mathcal{C}$ and $\mathcal{C}'$ small. Then precomposition with f defines a functor between the categories of presheaves*

$$(-) \circ f : Psh(\mathcal{C}'_\tau) \to Psh(\mathcal{C}_\tau).$$

Moreover, there is a geometric morphism between the categories of sheaves

$$(f^*, f_*) : Sh(\mathcal{C}'_\tau) \to Sh(\mathcal{C}_\tau)$$

where f_* is the restriction of $(-) \circ f$ to sheaves.

Proof. See [30]. □

Theorem 1.3.22 (Comparison Lemma). *Let $\mathcal{C}$, $\mathcal{C}$ be two small categories, and let τ (resp. τ') be a Grothendieck topology on $\mathcal{C}$ (resp. $\mathcal{C}'$). Let $u : \mathcal{C} \to \mathcal{C}'$ be a fully faithful functor that induces a morphism of Grothendieck topologies. If every object $X \in \mathcal{C}'$ has a covering $\{u(U_i) \to X\}_{i \in I}$ by objects of $\mathcal{C}$, then u induces an equivalence of categories of sheaves (of sets) $(u^*, u_*) : Sh(\mathcal{C}_\tau) \to Sh(\mathcal{C}'_{\tau'})$.*

Proof. See III.4.1. of [12] □

We introduce the notion of a topology in a topos. For more details, see [12], [33], [10], and Chapter 7, part 1 of [31].

Definition 1.3.23. *Let $\mathcal{C}$ be a category, and let τ and τ' be two Grothendieck topologies on $\mathcal{C}$. The **intersection topology** $\tau \cap \tau'$ of topologies τ and τ' is the finest topology on $\mathcal{C}$ that is coarser than both τ and τ'.*

In the sense of [12] (SGA4 IV.9), an embedding is defined as follows.

Definition 1.3.24. *Let $f : Sh(\mathcal{C}_\tau) \to Sh(\mathcal{C}_{\tau'})$ be a morphism of topoi. We say that f is a **embedding** if and only if it is fully faithful.*

Example 1.3.25. *Let $\mathcal{C}$ be a category, and let τ, τ' be its topologies. Let $\tau \cap \tau'$ denote the intersection topology of τ and τ'. The canonical morphisms of topologies induce geometric morphisms*

$$j = (j^*, j_*) : Sh(\mathcal{C}_\tau) \to Sh(\mathcal{C}_{\tau\cap\tau'}) \quad \text{and} \quad i = (i^*, i_*) : Sh(\mathcal{C}_{\tau'}) \to Sh(\mathcal{C}_{\tau\cap\tau'}).$$

Moreover, the right adjoins j_* and i_* are embeddings.

Definition 1.3.26. *[12] (SGA4)*

1. *The subtopos $E \subseteq Sh(\mathcal{C})$ is **open** if there exists a subsheaf F of the final object of $Sh(\mathcal{C})$ such that $E \cong Sh(\mathcal{C}/F)$.*

2. *The subtopos $E' \subseteq Sh(\mathcal{C})$ is **closed** if there exists a subsheaf F of the final object of $Sh(\mathcal{C})$ such that*

$$E' \cong \{G \in Sh(\mathcal{C}) : pr_1 : F \times G \to F \text{ is an isomorphism})\}.$$

3. *They are **complement** for each other if they are defined by the same F.*

Remark 1.3.27. *if F is a subsheaf of the final object of $Sh(\mathcal{C})$, then the topoi $Sh(\mathcal{C}/F)$ and $\{G \in Sh(\mathcal{C}$: $F \times G \to F$ is an isomorphism$)\}$ are subtopoi of $Sh(\mathcal{C})$ (See 7.43 of [31]).*

Remark 1.3.28. *The term "open" in the above definition refers to the open sets in the Lawvere-Tierney topology.*

Definition 1.3.29. *Let $f : Sh(\mathcal{C}_\tau) \to Sh(\mathcal{C}_{\tau'})$ be an embedding. We say f is an **open (resp. closed) immersion** if the essential image of f is an open (resp. closed) subtopos.*

Example 1.3.30. *If X is a topological space, then topos $Sh(X)/F$ is equivalent to $Sh(U)$ for some open subset U of X.*

Notation 1.3.31. *Denote by $\epsilon_{\tau'}$ the composition*

$$\mathcal{C}_\tau \xrightarrow{h_-} Psh(\mathcal{C}_\tau) \xrightarrow{a_\tau} Sh(\mathcal{C}_\tau)$$

$$\begin{array}{ccc} a & & hom(-,a) \\ \downarrow f & & \downarrow f\circ - \\ b & & hom(-,b) \end{array}$$

where a_τ is the sheafification with respect to topology τ.

Proposition 1.3.32. *Let $\mathcal{C}$ be a category, and let τ, τ' be its topologies. Let $\tau \cap \tau'$ denote the intersection topology of τ and τ'. Let $j = (j^*, j_*) : Sh(\mathcal{C}_\tau) \to Sh(\mathcal{C}_{\tau\cap\tau'})$ and $i = (i^*, i_*) : Sh(\mathcal{C}_{\tau'}) \to Sh(\mathcal{C}_{\tau\cap\tau'})$ be the canonical geometric morphisms induced by "inclusion" maps. The following properties are equivalent:*

1. *j_* makes $Sh(\mathcal{C}_\tau)$ an open subtopos of $Sh(\mathcal{C}_{\tau\cap\tau'})$, while i_* makes $Sh(\mathcal{C}_{\tau'})$ its closed complement;*
2. *Every object $X \in \mathcal{C}$ has a covering sieve R with respect to τ such that, for every $U \to X$ in R, the empty sieve is τ'-covering for U.*

Proof. Let $\varnothing_{\tau'}$ be the initial object of $Sh(\mathcal{C}_{\tau'})$. By [12] (SGA4.II.4.6.1), the empty sieve is τ'-covering for $U \in \mathcal{C}$ if and only if $\epsilon_{\tau'}(U) = \varnothing_{\tau'}$.

Write $W := i_*(\varnothing_{\tau'}) = i_*(a_{\tau'}(\varnothing))$, by the definition of sheafification, we have

$$W(X) = \begin{cases} * & \text{if } \epsilon_{\tau'}(X) = \varnothing_{\tau'}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Therefore W is a subsheaf of the final sheaf $*$.

(2) $\Rightarrow$ (1) : Firstly, we show that j is isomorphic to the embedding corresponding to the open subtopos $Sh(\mathcal{C}_{\tau\cap\tau'})/W$. For $F \in Sh(\mathcal{C}_{\tau\cap\tau'})$, the τ-sheaf j^*F is the τ-sheaf associated to the presheaf F. If $X \in \mathcal{C}$ satisfies $\epsilon_{\tau'}(X) = \emptyset_{\tau'}$, then $\epsilon_{\tau'}(U) = \emptyset_{\tau'}$ holds also for all $U \to X$, since the initial object of the topos is strict and the functor $\epsilon_{\tau'}$ induces a morphism of sheaves $\epsilon_{\tau'}(U) \to \epsilon_{\tau'}(X) = \emptyset_{\tau'}$. Therefore, every τ-covering sieve of U is also a $\tau \cap \tau'$-covering, consequently,

$$(j^*F)(X) = F(X) \quad \text{if } \epsilon_{\tau'}(X) = \emptyset_{\tau'}. \tag{1.1}$$

Hence, the map $W \times j_*j^*F$ is a $\tau \cap \tau'$-sheaf given by

$$X \mapsto \begin{cases} F(X) & \text{if } \epsilon_{\tau'}(X) = \emptyset_{\tau'}, \\ \varnothing & \text{otherwise.} \end{cases}$$

So if $F \in Sh(\mathcal{C}_{\tau\cap\tau'})$ has $Psh(F, W) \neq \varnothing$, then $F \to W \times j_*j^*F$ is an isomorphism. Therefore, every $F \in Sh(\mathcal{C}_{\tau\cap\tau'})/W$ satisfies $F \cong W \times j_*j^*F$.

Note that for each $U \in \mathcal{C}$, by equality 1.1, if $\epsilon_{\tau'}(U) = \emptyset_{\tau'}$, we have $j^*W(U) = W(U) = *$. By the hypothesis (2), every $X \in \mathcal{C}$ has a covering sieve R with respect to τ such that, for every $U \to X$ in R, the empty sieve is τ'-covering for U, so $j^*W(U) = *$ for every $U \in \mathcal{C}/R$. Applying Proposition 1.3.11, we obtain

$$j^*W(X) = hom(h_X, j^*W) = hom(R, j^*W) = \varprojlim_{(U\to X)\in\mathcal{C}/R} j^*W(U) = \varprojlim_{(U\to X)\in\mathcal{C}/R} * = *.$$

Therefore, $j^*W = *$ is the final sheaf. Hence for every $A \in Sh(\mathcal{C}_\tau)$ we have $j^*(W \times j_*A) = A$. Then the restriction of j^* to $Sh(\mathcal{C}_{\tau\cap\tau'})/W$ is an equivalence of categories from $Sh(\mathcal{C}_{\tau\cap\tau'})/W$ to $Sh(\mathcal{C}_\tau)$, a quasi-inverse being given by $A \mapsto W \times j_*A$.

In other words, j_* is isomorphic to the open embedding corresponding to the open subtopos $Sh(\mathcal{C}_{\tau\cap\tau'})/W$.

Now, we will prove that i_* is isomorphic to the closed complement embedding of j_*. The closed complement of $Sh(\mathcal{C}_{\tau\cap\tau'})/W$ is the full subcategory

$$\begin{aligned} E := &\{F \in Sh(\mathcal{C}_{\tau\cap\tau'}) : pr_1 : W \times F \to W \text{ is an isomorphism}\} \\ = &\{F : F(X) = * \text{ for all } X \text{ with } \epsilon_{\tau'}(X) = \emptyset_{\tau'}\} \end{aligned}$$

The functor i_* takes values in E, since for any $B \in Sh(\mathcal{C}_{\tau'})$, we have

$$(i_*B)(X) = hom(h_X, B) = hom(\epsilon_{\tau'}(X), B) \quad (X \in \mathcal{C}).$$

Hence, $i_*(Sh(\mathcal{C}_\tau)) \subseteq E$. It remains to prove that $i_*(Sh(\mathcal{C}_\tau)) \supseteq E$, or in other words, that every $F \in E$ is a τ'-sheaf.

Let $F \in E$ and $X \in C$, and let R' be a τ'-covering sieve of X. It is sufficient to show that $F(X) \cong Hom(R', F)$ (the definition of sheaf). By the hypothesis (2) there is a τ-covering sieve R of X consisting of objects which are covered by empty sieve under τ'. Now $R \cup R'$ is a covering sieve for $\tau \cap \tau'$. Since F is a sheaf for $\tau \cap \tau'$, the upper horizontal map in the following diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & hom(R \cup R', F) = \varprojlim_{(U \to X) \in C/(R \cup R')} F(U) \\ & \searrow & \downarrow \\ & & hom(R', F) = \varprojlim_{(U \to X) \in C/R'} F(U) \end{array}$$

is bijective. Since $F \in E$, we have $F(U) = *$ for each $(U \to X) \in C/R$. This implies that the right hand vertical map between the $\varprojlim$'s is bijective. Hence so is the diagonal map, as desired.

$(1) \Rightarrow (2)$: See (2.2) of [27]. □

1.4 Galois cohomology

In the first half of the 20th century, the concept of group cohomology was well developed, focusing on invariants associated with a group G equipped with a G-module S. This provided a purely algebraic tool for studying group representations, drawing analogies with the topological case. A notable application was in the context of Galois groups, leading to the notion of Galois cohomology, which became an important tool in the modern development of number theory.

In this section, we present some fundamental concepts of group cohomology, profinite groups, and Galois cohomology, with the primary references being [25] and [24].

Definition 1.4.1. *A* ***topological group*** *G is a topological space that is also a group such that the group operation*

$$\begin{aligned} \cdot : G \times G &\longrightarrow G \\ (a, b) &\longmapsto ab, \end{aligned}$$

and the inverse map

$$^{-1} : G \longrightarrow G \tag{1.2}$$

$$a \longmapsto a^{-1} \tag{1.3}$$

are continuous.

A morphism of topological groups is a continuous group homomorphism.

Definition 1.4.2. *Given a topological group G, a G-**set** is a set S equipped with a G-**action** on S, i.e., a function $\cdot : G \times S \to S$, $(g, s) \mapsto g \cdot s$ satisfies the following two axioms:*

- $e \cdot s = s$;
- $g \cdot (h \cdot s) = gh \cdot s$.

In this case, the topology doesn't matter.

*A **continuous** G-**sets** S is a set S equipped with an G-action $G \times S \to S$ that is continuous when S is given the discrete topology.*

*A morphism of G-sets is a function $f : S \to T$ such that $g \cdot f(s) = f(g \cdot s)$ for every $g \in G$ and $s \in S$. This morphism is called G-**equivariant map**.*

*The category of G-sets (resp. continuous G-sets), denoted G-**Set** ($\mathcal{C}G$-**Set**), is a category consists of G-sets (resp. continuous G-sets), with morphisms given by G-equivariant maps.*

Definition 1.4.3. *Given a topological group G, A G-**module** (resp. **continuous** G-**module**) is a G-set (resp. continuous G-set) M equipped with an abelian group operation, denoted $(M,+)$, such that*

$$g \cdot (x+y) = g \cdot x + g \cdot y$$

for every $g \in G$, $x, y \in M$.

A morphism of G-modules is a morphism of G-sets $f : M \to N$ which is also a group homomorphism.

*The category of G-modules (resp. continuous G-modules), denoted G-**Mod** ($\mathcal{C}G$-**Mod**), is a category that consists of G-modules (resp. continuous G-modules), with morphisms given by G-equivariant maps that are also group homomorphisms.*

Definition 1.4.4. *Let S be a G-set. For all $x \in S$, the **orbit** of x, denoted $G \cdot x$, is the set*

$$\{g \cdot x : g \in G\}.$$

*Given a subset $T \subseteq S$, the **stabilizer** of T, denoted $Stab(T)$, is the set*

$$\{g \in G : g \cdot x = x \ \ \forall x \in T\}.$$

We write $Stab(x)$ instead of $Stab(\{x\})$. it is immediate that $Stab(T)$ is a subgroup of G.

*Given a subset $L \subseteq G$, the L-**invariant of** S, denoted S^L, is the set*

$$\{x \in S : g \cdot x = x \ \ \forall g \in L\}.$$

Remark 1.4.5. *If S is a G-module, S^L is a subgroup of S.*

If $f : S \to T$ is a morphism of G-set, then f preserves invariant element, i.e., $Stab(x) \subseteq Stab(f(x))$ for all $x \in X$. In particular, if $x \in S^L$ for a $L \subseteq G$, then $f(x) \in T^L$, in other words, $f(S^L) \subseteq T^L$.

A G-set S is continuous if and only if the subgroup $Stab(x)$ is open subset of G for each $x \in S$.

Definition 1.4.6. *Let L be a subset of a topological group G, we define the L-**invariant functor** as*

$$\begin{aligned} (\ \ )^L : G\text{-}\textbf{Set} &\longrightarrow \textbf{Set} \\ S &\longmapsto S^L, \end{aligned}$$

and for a morphism $f : S \to T$ of G-sets, f^L is the induced restriction, as noted in the previous remark..

*We will use the same name and notation for the analogous functors $\mathcal{C}G$-**Set** $\to$ **Set**, G-**Mod** $\to$ **Ab**, and $\mathcal{C}G$-**Mod** $\to$ **Ab**.*

Proposition 1.4.7. *Given a topological group G, the category $\mathcal{C}G$-**Mod** is abelian and has sufficiently many injectives. Moreover, the G-invariant functor*

$$(\ )^L : \mathcal{C}G\text{-}\textbf{\textit{Mod}} \longrightarrow \textbf{\textit{Set}}$$

is left exact.

Proof. See [31], Lem. 19.3.1. □

From this definition, we can define the group cohomology.

Definition 1.4.8. *Given a topological group G, a G-module S, and a $q \geq 0$, we define **the q-th (group) cohomology of** G, with coefficients in the S by*

$$H^q(G, S) = R^q(\ )^G(S).$$

Here is the famous theorem about group cohomology.

Theorem 1.4.9 (Hilbert's Theorem 90). *If L/K is a finite Galois extension of fields with Galois group $G = Gal(L/K)$, then the first cohomology group of G, with coefficients in $L^\times$, is trivial:*

$$H^1(G, L^\times) = \{1\}.$$

A particularly important case for us is Galois cohomology, which is defined by taking G to be the absolute Galois group of a field, i.e., $Gal(k^{sep}/k)$, where k^{sep} denotes the separable closure of k, i.e., the maximal Galois extension of k. To define Galois cohomology, we first need to equip G with a topology. While it would be possible to use the discrete topology, this is not convenient for our purposes. Instead, we adopt the profinite topology on $Gal(k^{sep}/k)$, which is natural for the study of such Galois groups.

Definition 1.4.10. *A profinite group is a topological group that is isomorphic to the inverse limit (or projective limit) of a system (cofiltered diagram) of discrete finite groups.*

Since the cofiltered limit of a system of discrete finite groups can be viewed as a "subspace" of the product of these groups, we can equip it with the subspace topology inherited from the product topology on the Cartesian product of the finite groups in the system.

There are some useful properties of profinite groups.

Theorem 1.4.11. *Let G be a topological group. Then the following are equivalent:*

1. *G is profinite.*

2. *G is compact, Hausdorff, and totally disconnected.*

3. *The identity* 1 *admits a local base* $\mathcal{U}$ *such that each* $U \in \mathcal{U}$ *is an open normal subgroup of* G *with finite index and*

$$G = \varprojlim_{U \in \mathcal{U}} G/U.$$

Proof. See Theorem 2.1.3 of [25] □

Proposition 1.4.12. *Let* G *be a profinite group and* $\mathcal{U}$ *be the set of all open normal subgroups of* G*. Then* $\bigcap_{H \in \mathcal{U}} H = \{1\}$.

Proof. See Theorem 2.1.3 of [25]. □

Proposition 1.4.13. *Let* G *be a profinite group, and let* $\mathcal{U}$ *be the set of all open normal subgroups of* G*. If* A *is a* G*-set equipped with discrete topology, then* A *is a continuous* G*-set (i.e., the action is continuous) if and only if* $A = \bigcup_{U \in \mathcal{U}} A^U$.

Proof. Assume A is a continuous G-set, then $stab(x)$ is an open subgroup of G for each $x \in A$. By the previous proposition, for any $x \in A$ there is an open normal subgroup H contained in $stab(x)$. Therefore, $x \in A^{stab(x)} \subseteq A^H$, as desired.

Assume $A = \bigcup_{U \in \mathcal{U}} A^U$, we need to prove that the action $f : G \times A \to A$ is continuous. Since the topological space A is discrete, it suffices to show that the preimage $f^{-1}(b)$ is an open subset, for every $b \in A$. Fix a $b \in A$, since $A = \bigcup_{U \in \mathcal{U}} A^U$, there exists an open normal subgroup H such that $b \in A^H$. Thus, for any $(g, a) \in f^{-1}(b)$, $Hg \times \{a\}$ is an open subset of (g, a) such that $Hg \times \{a\} \subseteq f^{-1}(b)$. Therefore, $f^{-1}(b)$ is an open subset as desired. □

The following theorem shows that the Galois group is a profinite group.

Theorem 1.4.14. *For any field* F*, and any Galois extension* K

$$Gal(K/F) = \varprojlim_{L} Gal(L/F),$$

where L *runs over finite Galois extensions of* F*, such that* $F \subseteq L \subseteq K$*. Furthermore, the topology given by profinite group agree with the* ***Krull topology****, i.e., the topology generated by the local base* $U_i := Gal(K/L_i)$ *where* L_i *is a finite Galois extension of* F*.*

Proof. See Theorem 2.11.1 of [25]. □

The Fundamental Theorem of Galois Theory can now be formulated in terms of profinite groups.

Theorem 1.4.15. *Let K/F be a Galois extension with Galois group $G := Gal(K/F)$. We denote the set of intermediate fields $F \subseteq L \subseteq K$ by $I(K/F)$, and the set of closed subgroups of G by $S(G)$. Then, there is a bijection between $I(K/F)$ and $S(G)$ as follows:*

$$\phi : I(K/F) \longrightarrow S(G)$$
$$L \longmapsto Gal(K/L).$$

Its inverse is

$$\psi : S(G) \longrightarrow I(K/F)$$
$$H \longmapsto K^H$$

where K^H denotes the fixed subfield of K under H.

Proof. See Theorem 2.11.3 of [25]. □

Remark 1.4.16. *In a topological group, every open subset is also closed, ensuring that the function is well-defined.*

Definition 1.4.17. *Let K be a field with separable closure k^{sep}, and let $G = Gal(k^{sep}/k)$ be a topological group equipped with profinite topology. In this case, $H^q(G, S)$ is called **the q-th Galois cohomology** of G on S.*

Let G be a profinite group. We denote the canonical topology of $\mathcal{C}G$-**Set** by T_G. It is easy to check that a family $\{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ of morphisms in $\mathcal{C}G$-**Set** is a family of universal effective epimorphisms if and only if $U = \bigcup_{i \in I} \phi_i(U_i)$.

Proposition 1.4.18. *Let G be a profinite group. The functor*

$$\phi : \mathcal{C}G\text{-}\mathbf{Set} \longrightarrow Sh(\mathcal{C}G\text{-}\mathbf{Set}, T_G)$$
$$Z \longmapsto hom_G(-, Z)$$

*is an equivalence of categories, where $hom(X, Z)$ denotes the set of all morphisms $X \to Z$ in the category $\mathcal{C}G$-**Set**. The quasi-inverse of the functor ϕ is the functor*

$$\psi : Sh(\mathcal{C}G\text{-}\mathbf{Set}, T_G) \longrightarrow \mathcal{C}G\text{-}\mathbf{Set}$$
$$F \longmapsto \varinjlim_{H} F(G/H).$$

Here for every open normal subgroup H of G, the quotient group G/H is viewed as a continuous G-module via left multiplication. We define a continuous G-structure on the set $F(G/H)$ as follows:

For $g \in G/H$ and $s \in F(G/H)$ we set $gs = F(R_g)(s)$, where $R_g : G \to G$ is the G-map given by $g' \mapsto g' \cdot g$.

Moreover, if H and H' are both open normal subgroups of G with $H \subseteq H'$, the canonical G-homomorphism $G/H \to G/H'$ induces a map $F(G/H) \to F(G/H')$. The inductive limit is taken over all open normal subgroups of G, ordered by inclusion. Thus, $\varinjlim_H F(G/H)$ has a natural structure of a continuous G-sets.

Proof. On the one hand, $\psi \circ \phi(Z) = \varinjlim_H hom_G(G/H, Z)$ for each object $Z \in \mathcal{C}G$-**Set**. We have a canonical identification

$$\varinjlim_H hom_G(G/H, Z) = \varinjlim_H Z^H = \bigcup_H Z^H = Z.$$

The first equality holds because a G-map $\phi : G/H \to Z$ is completely determined by the value of $\phi(1) \in Z^H$. The second equality follows from the fact that $Z^H \cap Z^{H'} = Z^{HH'}$. The third equality follows from the definition of a continuous G-set, which is based on the equivalence criterion for continuity. On the other hand,

$$\phi \circ \psi(F) = hom_G(-, \varinjlim_H F(G/H)).$$

Now, it suffices to prove that there is an isomorphism of sheaves from F to $hom_G(-, \varinjlim_H F(G/H))$, functorial in F.

Let U be a continuous G-set. Since $U = \bigcup_H U^H$, the family $\{U^H \to U\}$ of all inclusions $U^H \hookrightarrow U$ is a covering in the topology T_G. So, we obtain an exact diagram

$$F(U) \to \prod_H F(U^H) \rightrightarrows \prod_{H,H'} F(U^H \times_U U^{H'}).$$

Note that $U^H \times_U U^{H'} = U^H \cap U^{H'} = U^{HH'}$, then we have a canonical identification

$$F(U) = \{(s^H) \in \prod F(U^H) : s^H|_{U^{HH'}} = s^{H'}|_{U^{HH'}}\} = \varprojlim F(U^H).$$

It's easy to see that the family $\{G/H \xrightarrow{\varphi_u} U^H\}_{u \in U^H}$, where $\varphi_u(gH) = gu$, is a covering in the topology T_G (here, φ_u is well defined since u is H-invariant). Then, For a sheaf F we have the exact diagram

$$F(U^H) \to \prod_{u \in U^H} F(G/H) \rightrightarrows \prod_{u,v \in U^H} F(G/H \times_{U^H} G/H)$$

corredsponding to the covering. This shows that the image of the injective map

$$F(U^H) \to \prod_{u \in U^H} F(G/H) = hom(U^H, F(G/H))$$

is precisely the subset $hom_{G/H}(U^H, F(G/H))$ of G/H-maps from U^H to $F(G/H)$: In details, the map $F(U^H) \to \prod_{u \in U^H} F(G/H)$ is given by

$$\begin{aligned} \Psi : F(U^H) &\longrightarrow \prod_{u \in U^H} F(G/H) = hom(U^H, F(G/H)) \\ \alpha &\longmapsto \Psi(\alpha) \end{aligned}$$

with

$$\begin{aligned} \Psi(\alpha) : U^H &\longrightarrow F(G/H) \\ u &\longmapsto F(\varphi_u)(\alpha). \end{aligned}$$

The map $\Psi(\alpha)$ is a G/H-map, since for each $gH \in G/H$, we have

$$gH \cdot \Psi(\alpha)(u) = gH \cdot F(\varphi_u)(\alpha) = F(R_{gH})F(\varphi_u)(\alpha) = F(\varphi_u \circ R_{gH})(\alpha) = F(\varphi_{gu})(\alpha) = \Psi(\alpha)(gHu).$$

This implies $\mathrm{Im}(\Psi) \subseteq hom_{G/H}(U^H, F(G/H))$. To prove the other inclusion, assume that $r = (r_u)_{u \in U^H}$ is G/H-equivariant, i.e.,

$$gH \cdot r_u = F(R_g) r_u = r_{g \cdot u}$$

for each $g \in G$. we want to show that for any $u, v \in U^H$,

$$F(p_1)(r_u) = F(p_2)(r_v) \in F(\{(jH, kH) \in G/H \times G/H : j \cdot u = k \cdot v\}).$$

Let $E_{u,v} = \{fH \in G/H : f \cdot u = v\}$. And consider the map

$$\begin{aligned} G/H \times E_{u,v} &\longrightarrow G/H \times_{U^H} G/H \\ (jH, kH) &\longmapsto (jkH, jH) \end{aligned}$$

From the definition of $E_{u,v}$, this map is well-defined, and in fact, consider $G/H \times E_{u,v}$ as the disjoint union $\coprod_{f\in E_{u,v}} G$, this map is G/H-equivariant, and is component-wise given by the maps

$$\begin{aligned} i_f : G/H &\longrightarrow G/H \times_{U^H} G/H = \{(jH, kH) : j \cdot u = kf \cdot u\} \\ gH &\longmapsto (gfH, gH). \end{aligned}$$

The map $G/H \times E_{u,v} \longrightarrow G/H \times_{U^H} G/H$ has an inverse $(jH, kH) \mapsto (kH, k^{-1}jH)$. The map $(jH, kH) \mapsto (kH, k^{-1}jH)$ is well-defined (i.e., $(kH, k^{-1}jH) \in G/H \times E_{u,v}$) since

$$k^{-1}j \cdot u = k^{-1}k \cdot v = v.$$

This map is G/H-equivariant, because of

$$(fjH, fkH) \mapsto (fkH, k^{-1}f^{-1}fjH) = (fkH, k^{-1}jH) = fH \cdot (kH, k^{-1}jH).$$

So, the map $G/H \times E_{u,v} \longrightarrow G/H \times_{U^H} G/H$ is an isomorphism. Because F is a sheaf, $F(\coprod A) = \prod F(A)$, in particular,

$$F(G/H \times_{U^H} G/H) \cong F(G/H \times E_{u,v}) \cong \prod_{f\in E_{u,v}} F(G/H)$$

So by construction an element $x \in F(G/H \times_{U^H} G/H)$ maps to the family $(F(i_f)(x))_{f\in E_{u,v}}$. Since this is an isomorphism, to prove $F(p_1)(r_u) = F(p_2)(r_v)$, it is sufficient to prove this equality after applying $F(i_f)$ for all $f \in E_{u,v}$. Before computing, note that

$$p_\lambda \circ i_f(gH) = p_\lambda(gfH, gH) = \begin{cases} gfH, & \text{if } \lambda = 1 \\ gH, & \text{if } \lambda = 2, \end{cases}$$

i.e., $p_1 \circ i_f = R_{fH}$ and $p_2 \circ i_f = id_{G/H}$. Now, we obtain

$$\begin{aligned} F(i_f)F(p_1)(r_u) =& F(p_1 \circ i_f)(r_u) = F(R_{fH})(r_u) = r_{fu} = r_v \\ =& F(id_{G/H}(r_v)) = F(p_2 \circ i_f) = F(i_f)F(p_2)(r_v). \end{aligned}$$

So, from the sheaf condition, the image of map $F(U^H) \to \prod_{u\in U^H} F(G/H) = hom(U^H, F(G/H))$ is precisely the subset $hom_{G/H}(U^H, F(G/H))$. This map

$$F(U^H) \to hom_{G/H}(U^H, F(G/H))$$

is functorial in U^H, so is an isomorphism of sheaves, and it is functorial in F.

We want to show next that the map $F(G/H) \to \varinjlim_{H'} F(G/H')$ induces a canonical isomorphism

$$hom_{G/H}(U^H, F(G/H)) \to hom_G(U^H, \varinjlim_{H'} F(G/H')).$$

To show this, note that given a normal subgroup $H' \subseteq H$, the family $\{\pi : G/H' \to G/H\}$ is a covering in T_G. So there is an associated exact diagram

$$F(G/H) \to F(G/H') \rightrightarrows F(G/H' \times_{G/H} G/H').$$

We use the previous strategy to show that the image of the map $F(G/H) \to F(G/H')$ is the subset $F(G/H')^{H/H'}$ of H/H' invariant elements in $F(G/H')$: Let's prove that for any $x \in F(G/H)$, $F(\pi)(x)$ is H/H'-invariant first. Let hH' be any element in H/H', we have

$$hH'F(\pi)(x) = F(R_{hH'})F(\pi)(x) = F(\pi \circ R_{hH'})(x),$$

but $\pi \circ R_{hH'}(gH') = \pi(ghH') = gH \cdot hH = gH = \pi(gH')$, thus

$$hH'F(\pi)(x) = F(\pi \circ R_{hH'})(x) = F(\pi)(x).$$

As before, to show that any H/H'- invariant element y is an element of the image of the map $F(G/H) \to F(G/H')$, it is sufficient to prove that $F(p_1)(y) = F(p_2)(y)$. Firstly, We note that

$$G/H' \times_{G/H} G/H' = \{(jH', kH') \in G/H' \times G/H' : jH = kH\}$$

and the map

$$\coprod_{hH' \in H/H'} G/H' = G/H' \times H/H' \longrightarrow G/H' \times_{G/H} G/H'$$
$$(gH', hH') \longmapsto (gH', ghH')$$

is an isomorphism of G/H'-sets with inverse

$$(jH', kH') \mapsto (jH', j^{-1}kH').$$

It is easy to check that those maps are well-defined and are G/H' equivariant. Because F is a sheaf, we have

$$F(G/H' \times_{G/H} G/H') \cong \prod_{hH' \in H/H'} F(G/H').$$

So by construction an element $x \in F(G/H' \times_{G/H} G/H')$ maps to the family $(F(i_{hH'})(x))_{hH' \in H/H'}$, where $i_{hH'}$ is a G/H'-map from G/H' to $G/H' \times_{G/H} G/H'$ given by $g \mapsto (gH', ghH')$. Since this is an isomorphism, to prove $F(p_1)(y) = F(p_2)(y)$, it is sufficient to prove this equality after applying $F(i_{hH'})$ for all $hH' \in H/H'$. Since $p_1 \circ i_{hH'} = id_{G/H'}$ and $p_2 \circ i_{hH'} = R_{hH'}$, we have

$$\begin{aligned} F(i_{hH'})F(p_1)(y) &= F(p_1 \circ i_{hH'})(y) = F(id_{G/H'})(y) = y = hH' \cdot y \\ &= F(R_{hH'})(y) = F(p_2 \circ i_{hH'}) = F(i_{hH'})F(p_2)(y) \end{aligned}$$

as desired. So the map $F(G/H) \to F(G/H')$ identifies the set $F(G/H)$ with the set of H/H'-invariant elements in $F(G/H')$.

Therefore the map $F(G/H) \to \varinjlim_{H'} F(G/H')$ identifies the set $F(G/H)$ with the subset $(\varinjlim_{H'} F(G/H'))^H$ of H-invariant elements in $\varinjlim_{H'} F(G/H')$. Hence the map

$$hom_{G/H}(U^H, F(G/H)) \to hom_G(U^H, \varinjlim_{H'} F(G/H')).$$

is in fact an isomorphism.

Putting all together, we obtain the canonical isomorphisms

$$\begin{aligned} F(U) &= \varprojlim_{H} F(U^H) \\ &\cong \varprojlim_{H} hom_{G/H}(U^H, F(G/H)) \\ &\cong \varprojlim_{H} hom_G(U^H, \varinjlim_{H'} F(G/H')) \\ &\cong hom_G(\varinjlim_{H} U^H, \varinjlim_{H'} F(G/H')) \\ &\cong hom_G(U, \varinjlim_{H'} F(G/H')) \end{aligned}$$

which are functorial both in U and in F. This completes the proof of the proposition. □

This implies

Corollary 1.4.19. *The category $\mathcal{C}G$-**Mod** is equivalent to the category $Ab(\mathcal{C}G\text{-}\mathbf{Mod}, T_G)$. The equivalence is given by the mutually quasi-inverse functors $A \mapsto hom_G(-, A)$ and $F \mapsto \varinjlim_H F(G/H)$.*

In the final part of this work, these theorems will be used to explore the relationship between Galois cohomology and étale cohomology.

Chapter 2

Étale site and its sheaves

This chapter focuses on the main subject of the thesis: the étale site. In the first section, we summarize key results from commutative algebra needed to define étale morphisms of schemes and present some essential properties of these morphisms to support the continuation of our work.

In the second section, we define étale sheaves, prove a criterion for checking whether a presheaf is a sheaf, show that any representable presheaf is a sheaf, and provide several classical examples of étale sheaves.

At the end of the chapter, we introduce the direct image functor and inverse image functor between categories of sheaves on étale sites, which are essential tools for studying the relationship between categories of sheaves on two different étale sites.

2.1 Étale morphism

An étale morphism is the analogue in algebraic geometry of local homeomorphism in topology and a covering of Riemann surfaces with no branch point in complex analysis.

Definition 2.1.1. *A ring homomorphism $\psi : A \to B$ is **flat** if the tensor product functor from A-modules to B-modules given by $M \mapsto B \otimes_A M$ is exact. One also says that B is a flat A-algebra.*

Remark 2.1.2. *The tensor product functor is always right exact functor. This definition is equivalent to: for every injective linear map $\phi : M \to N$ of A- modules, the map*

$$\begin{aligned} B \otimes_A \phi : B \otimes_A M &\longrightarrow B \otimes_A N \\ b \otimes k &\longmapsto b \otimes \phi(k). \end{aligned}$$

is also injective.

From this characterization, the map $B \otimes_A id_A$ is injective, in particular, the map $1 \otimes a \mapsto \psi(a) \otimes 1$ is injective. Hence, flat map ψ is always injective.

The flatness is a local property.

Proposition 2.1.3. *If $\psi : A \to B$ is flat, then the following statements are equivalent:*

1. *B is a flat A-module;*

2. *$B_{\mathfrak{p}}$ is a flat $A_{\psi^{-1}(\mathfrak{p})}$-module for every prime ideal $\mathfrak{p}$ of B;*

This motivates the definition of a flat morphism of schemes.

Definition 2.1.4. *A morphism $\psi : Y \to X$ of schemes is* ***flat*** *if the local homomorphisms $\mathcal{O}_{X,\psi(Y)} \to \mathcal{O}_{Y,y}$ are flat for all $y \in Y$.*

Remark 2.1.5. *A flat morphism $\phi : Y \to X$ of varieties is the analogue in algebraic geometry of continuous family of manifolds $Y_x = \phi^{-1}(x)$ parametrized by the points of X in differential topology. If ϕ is flat then,*

$$\dim \phi^{-1}(x) = \dim Y - \dim X$$

for all $x \in X$ with $Y_x \neq \varnothing$. This resembles the preimage theorem in differential topology.

Here are some well-known facts about flat ring maps that will be useful for developing the next part of the work.

Remark 2.1.6.

- *Let $\psi : A \to B$ be a ring homomorphism. If B is a free A-module, then ψ is flat.*
- *Let R be a ring. And let $S \subseteq R$ be a multiplicative subset. Then the localization $S^{-1}R$ is a flat R-algebra (see Lemma 10.39.18. Part I of [31]).*
- *A composition of flat ring maps is a flat.*

Definition 2.1.7. *A local homomorphism $\phi : A \to B$ of local rings is* ***unramified*** *if B/m_B is finite separable field extension of A/m_A.*

Definition 2.1.8. *A locally of finite presentation morphism $f : Y \to X$ of schemes is* ***unramified*** *if the local homomorphisms $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ are unramified (i.e., $\kappa(y)$ is a finite separable field extension of $\kappa(f(y))$) for every $y \in Y$.*

Proposition 2.1.9. *Let $f : Y \to X$ be a morphism of schemes. The following properties are equivalent:*

1. *f is unramified.*

2. *For every $x \in X$, the x-fiber decomposes as $Y_x = \bigsqcup_{i \in I} Spec\ k_i$, where $k_i/\kappa(x)$ is a finite and separable field extension, for every $i \in I$.*

3. *f is locally of finite presentation, and the diagonal map $\Delta_f : Y \to Y \times_X Y$ is an open immersion.*

Proof. See Proposition 3.2 and 3.5 of [20]. □

Remark 2.1.10. *Let $f : Y \to X$ be a finite morphism between smooth connected affine curves over $\mathbb{C}$, and let y be a closed point of Y. We then have the local ring homomorphism $f^{\#} : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ Since $\mathcal{O}_{Y,y}$ is a discrete valuation ring, there is a unique integer $e_y > 0$ such that $f^{\#}(m_{f(y)})\mathcal{O}_{Y,y} = m_y^{e_y}$. This integer is called the ramification index or multiplicity of y over $f(y)$. Since $\kappa(y) = \kappa(f(y)) = \mathbb{C}$, f is unramified at y if and only if $e_y = 1$. So a flat unramified morphism is the analogue in algebraic geometry of a covering of Riemann surfaces with no branch point in complex analysis.*

Definition 2.1.11. *A morphism $f : Y \to X$ of schemes is **étale** if it is flat and unramified. A homomorphism $A \to Y$ of rings is **étale** if the corresponding morphism of schemes is étale.*

This definition of an étale morphism may seem abstract at first glance, but there is a more concrete description of étale morphisms that can make them easier to understand.

Proposition 2.1.12. *Let A be a ring, and let $f(x) \in A[x]$ be a monic polynomial. If $b \in A[x]/(f(x))$, and if the derivate $f'(x)$ is invertible in $(A[x]/f(x))_b$, then the canonical homomorphism $A \to (A[x]/f(x))_b$ is étale. In this case, the algebra $(A[x]/f(x))_b$, is called **standard étale algebra**, the canonical homomorphism $i : A \to (A[x]/f(x))_b$ is called **standard étale homomorphism**.*

Proof. Since $f(x)$ is monic, $B := A[T]/f(T)$ is a free A-module, and in particular $A[T]/f(T)$ is a flat A-module. Moreover, since $(A[T]/f(T))_b$ is a localization of $(A[T]/f(T))$, it is flat B-module. Hence, $(A[T]/f(T))_b$ is a flat A-module. By Proposition 2.1.3, the corresponding morphism of schemes is flat.

Next, we show that the corresponding morphism of schemes is unramified. This morphism is locally of finite presentation, so we only need to prove that for any $\overline{\mathfrak{p}} \in Spec\ B$ with $b \notin \overline{\mathfrak{p}}$ (since any prime ideal of B_b corresponds to a prime ideal of B that does not contains b), the induced homomorphism

$$Frac(A/\mathfrak{q}) \to Frac((B_b)/\overline{\mathfrak{p}}B_b) = Frac(B/\overline{\mathfrak{p}})$$

of fields is a finite separable extension, where $\mathfrak{q} = i^{-1}(\overline{\mathfrak{p}}B_b)$.

Note that $\overline{\mathfrak{p}}$ is the image of a prime ideal $\mathfrak{p}$ in $A[x]$ under the projection map $A[x] \to B$, where $(f) \subseteq \mathfrak{p}$. Therefore, $\mathfrak{q} = \mathfrak{p} \cap A$. Since $\mathfrak{q}$ is a prime ideal, $\mathfrak{q}[x]$ is a prime ideal of $A[x]$ contained in $\mathfrak{p}$. Thus, there exists a homomorphism

$$j : (A/\mathfrak{q})[x] \cong A[x]/\mathfrak{q}[x] \longrightarrow A[x]/\mathfrak{p}.$$

By the isomorphism theorem, we have

$$(A/\mathfrak{q})[x]/\mathrm{Ker}(j) \cong A[x]/\mathfrak{p}.$$

Now, let K denote the field of fractions of $A/\mathfrak{q}$. By the same reasoning, we obtain $B/\overline{\mathfrak{p}} \cong A[x]/\mathfrak{p}$. From the identification $(A/\mathfrak{q})[x]/\mathrm{Ker}(j) \cong A[x]/\mathfrak{p}$, we obtain a sequence of homomorphisms

$$(A/\mathfrak{q})[x] \longrightarrow (A/\mathfrak{q})[x]/\mathrm{Ker}(j) \longrightarrow Frac(A[x]/\mathfrak{p}) \cong Frac(B/\overline{\mathfrak{p}}).$$

For any non-zero $a + \mathfrak{q} \in A/\mathfrak{q}$, the image of $a + \mathfrak{q}$ under the homomorphism

$$(A/\mathfrak{q})[x] \to Frac(A[x]/\mathfrak{p})$$

is non-zero, since $a \notin \mathfrak{q} = \mathfrak{p} \cap A$. Therefore, there is a homomorphism induced by the localization

$$\begin{array}{ccc} (A/\mathfrak{q})[x] & \longrightarrow & Frac(A[x]/\mathfrak{p}) \\ \downarrow & \overset{h}{\nearrow} & \\ K[x] = (A/\mathfrak{q} \setminus \{0\})^{-1}(A/\mathfrak{q})[x] & & \end{array}.$$

By the isomorphism theorem, there is an injective homomorphism

$$\tilde{h} : K[x]/\mathrm{Ker}(h) \to Frac(A[x]/\mathfrak{p}).$$

Now:

- Since $K[x]$ is a PID, the prime ideal $\mathrm{Ker}(h)$ is a maximal ideal, so $K[x]/\mathrm{Ker}(h)$ is a field;
- Since $(A/\mathfrak{q})[x] \to (A/\mathfrak{q})[x]/\mathrm{Ker}(j) \cong A[x]/\mathfrak{p}$ is surjective, we have $A[x]/\mathfrak{p} \subseteq \mathrm{Im}(\tilde{h})$.

Thus, since the field of fractions of $A[x]/\mathfrak{p}$ is the least field containing $A[x]/\mathfrak{p}$, $\tilde{h}$ is an isomorphism. That is,

$$K[x]/\mathrm{Ker}(h) \cong Frac(A[x]/\mathfrak{p}) \cong Frac(B/\overline{\mathfrak{p}}) \cong Frac(B_b/\overline{\mathfrak{p}}B_b).$$

Finally, it suffices to show that $K[x]/\mathrm{Ker}(h)$ is a finite separable extension. Let $\overline{f}, \overline{f'}$ denote the image of f and f' in $K[x]$. Since $\mathfrak{p} \supseteq (f)$, we have $\overline{f} \in \mathrm{Ker}(h)$. But $K[x]$ is a PID, there exists an irreducible factor f_0 of $\overline{f}$ such that $\mathrm{Ker}(h) = (f_0)$. Because $f'(x)$ is invertible in $(A[x]/f(x))_b$, it is also invertible in $K[x]/\mathrm{Ker}(h)$. The following equality:

$$(\overline{f})' = (f_0 f_1)' = f_0' f_1 + f_0 f_1'$$

shows that f_0' is invertible in $K[x]/\mathrm{Ker}(h)$, which implies that there exist $g_0, g_1 \in K[x]$ such that

$$g_0 f_0 + g_1 f_0' = 1.$$

Thus, $gcd(f_0, f_0') = 1$, which means f_0 is a separable polynomial in $K[x]$. Therefore $K[x]/\mathrm{Ker}(h)$ is a finite separable extension of $K = A/\mathfrak{q}$, as desired. □

Definition 2.1.13. *An étale morphism $f : Y \to X$ is said to be* ***standard*** *if it is isomorphic to the*

$$Spec\ (A[x]/f(x))_b \longrightarrow Spec\ A$$

where the induced ring homomorphism $A \to (A[x]/f(x))_b$ is a standard étale homomorphism.

Theorem 2.1.14. *For any étale morphism $f : Y \to X$ and $y \in Y$, there exist open affine neighborhoods V of y and U of $f(y)$ such that $f(V) \subseteq U$ and the restriction $f|_V : V \to U$ is a standard étale morphism.*

Proof. See Theorem 3.14 of [20]. □

There are some important properties of étale morphisms.

Proposition 2.1.15. 1. *Every open immersion is étale.*

2. *Every base change of an étale morphism is étale.*

3. *The composite of two étale morphisms is étale.*

4. *If $f \circ g$ and f are étale, then g is also étale,*

5. *Every étale morphism is quasi-finite and open.*

Proof. See section 2 (chapter I) of [21] or 29.36 of [31]. □

Now, we observe some immediate consequences of the previous proposition.

Corollary 2.1.16. 1. *The diagonal map of étale map is étale.*

2. *The kernel map of étale maps is étale.*

3. *If U is an open subset of a scheme X, then the inclusion $U \hookrightarrow X$ is étale.*

Proof. They are consequences of 2) and 1) of the previous proposition. □

Definition 2.1.17. *We denote by Et/X the category whose objects are the étale morphisms $U \to X$ and whose morphisms are the X-morphism $f : U \to V$. This category is referred to as the category of étale schemes over X (if $X = Spec\ A$, it is also called the category of étale schemes over A, denoted Et/A).*

From 4) of Proposition 2.1.15, we deduce that each morphism of Et/X is an étale morphism.

Proposition 2.1.18. *Let k be a field and X be a k-scheme. Then $X \to Spec\ k$ is étale if and only if we can write X as a disjoint union of spectrum of finite and separable field extensions of k.*

Proof. The proposition follows from Proposition 2.1.9 and from the fact that $- \otimes_k \mathcal{O}_{X,x}$ is exact for every $x \in X$ (flatness). □

Corollary 2.1.19. *The category Et/k of étale schemes over a field k admits arbitrary coproduct.*

2.2 Étale sheaf

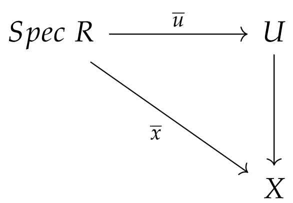
The site X_{et} that we are interested in is called the **étale site** on X. The underlying category of site X_{et} is Et/X, and the coverings of this site are the surjective families of étale morphisms $\{f_i : U_i \to U\}$ in Et/X, i.e., the families of étale morphisms satisfying $U = \bigcup f_i(U_i)$.

Since every open immersion is an étale morphism, the étale topology is finer than the Zariski topology. Consequently, the étale topology provides a cohomology theory that detects more information than the Zariski topology. For example, the cohomology defined on the étale topology with values in a constant sheaf can be non-trivial, whereas any constant sheaf in the Zariski topology is acyclic.

An **étale sheaf** is a sheaf on Et/X. And the **étale cohomology**, denoted $H^n_{et}(X, -)$, is the sheaf cohomology on X_{et}.

Here is an important notion for studying étale sheaves, strict localization, although we will not go into it in depth.

Definition 2.2.1. *A **geometric point** of a scheme X is a morphism $\overline{x} : Spec\ R \to X$ with R a separably closed field. An **étale neighborhood** of a geometric point $\overline{x} : Spec\ R \to X$ is an étale morphism $U \to X$ together with another geometric point $\overline{u} : Spec\ R \to U$ lying over $\overline{x}$, i.e., the diagram*



commutes.

The ***local ring at $\overline{x}$ for the étale topology*** *or* ***strictly local ring*** *at $\overline{x}$ is*

$$\mathcal{O}_{X,\overline{x}} := \varinjlim_{(U,\overline{u})} \Gamma(U, \mathcal{O}_U)$$

where the limit is over the connected affine étale neighborhoods $(U, \overline{u})$ of $\overline{x}$.

There is a criterion that simplifies checking whether a presheaf is an étale sheaf.

Proposition 2.2.2. *Let F be a presheaf on X_{et}. If F satisfies the sheaf condition for Zariski open coverings and for any étale coverings $(V \to U)$ (a single map) with V and U affine schemes, then F is an étale sheaf.*

Proof. We use the Corollary 1.3.14. So, given any covering $\{U_i \to U\}$ it suffices to find a refinement $\{V_j \to U\}$ which satisfies the sheaf condition.

Let $\{U_i \to U\}$ be an arbitrary covering in X_{et}. We choose an affine open covering

$$U = \bigcup_{j \in J} W_j$$

of U. This provides for each j a covering

$$U_i \times_U W_j \to W_j$$

in X_{et}. Let $\{U_{ijk}\}$ be an affine open cover of $U_i \times_U W_j$ for each i, j. Then, we obtain a refinement

$$\{U_{ijk} \to W_j\} \to \{U_i \times_U W_j \to W_j\}$$

of $\{U_i \times_U W_j \to W_j\}$. As an affine scheme, W_j is quasi-compact. Moreover, the map $U_{ijk} \to W_j$ is open, since they are étale morphisms. Therefore, we can refine the covering $\{U_{ijk} \to W_j\}$ by a finite subcovering $\{U_{jl} \to W_j\}$:

$$\{U_{jl} \to W_j\} \to \{U_{ijk} \to W_j\}.$$

If we compose the covering $\{U_{jl} \to W_j\}$ with the Zariski covering $\{W_j \to U\}$, we obtain a covering

$$\{U_{jl} \to U\}$$

together with a natural refinement map

$$\{U_{jl} \to U\} \to \{U_i \to U\}$$

If we show that $\{U_{jl} \to U\}$ satisfies the sheaf condition, then we prove the proposition. By construction, $\{U_{jl} \to U\}$ is a composite of $\{W_j \to U\}$ and $\{U_{jl} \to W_j\}$. So, if F satisfies the sheaf condition for $\{W_j \to U\}$ and all coverings $\{U_{jl} \to W_j\}$, then F satisfies the sheaf condition for the composite.

Note that

1. $\{W_j \to U\}$ is a Zariski covering;

2. $\{U_{jl} \to W_j\}$ is a finite family of morphisms of affine schemes.

So, from hypothesis, F satisfies the sheaf condition for $\{W_j \to U\}$.

Since $\{U_{jl} \to W_j\}$ is a finite family of morphisms of affine schemes, we can form an affine scheme $\coprod_l U_{jl}$, which allows us to write the covering $\{U_{jl} \to W_j\}$ as a composite of the coverings

$$\{U_{jl} \to \coprod_l U_{jl}\} \text{ and } \{\coprod_l U_{jl} \to W_j\}.$$

The left covering is a Zariski covering and the right covering is a family consisted of a unique morphism of affine schemes. by the hypothesis, F satisfies sheaf condition for both coverings, and so for the composite $\{U_{jl} \to W_j\}$. □

Recall that a flat homomorphism $f : A \to B$ is **faithfully flat** if it satisfies one of the following equivalent conditions:

1. If M is an A-module such that $M \otimes_R N = 0$, then $M = 0$.

2. A sequence of A-module

$$M' \to M \to M''$$

is exact if and only if the sequence obtained by tensoring over A with B

$$M' \otimes_A B \to M \otimes_A B \to M'' \otimes_A B$$

is exact.

3. The induced map $Spec\ B \to Spec\ A$ is surjective.

For this reason, a surjective étale morphism $(V \to U)$ of affine schemes corresponding to a faithfully flat homomorphism of rings $A \to B$. So, to check the second condition of above criterion, we shall usually make use only the faithfully flat ring homomorphism.

Example 2.2.3 (The structure sheaf)**.** *Let X be a scheme. For any étale morphism $U \to X$, define $\mathcal{O}_{X_{et}}(U) = \Gamma(U, \mathcal{O}_U)$. By Proposition 2.2.2, this is a sheaf on X_{et}. Obviously, its restriction to Zariski open coverings is a sheaf, so it suffices to establish the following result.*

Proposition 2.2.4. *For every faithfully flat homomorphism $f : A \to B$, the sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{i_2 - i_1} B \otimes_A B$$

is exact, where $i_2 - i_1$ is given by $b \mapsto 1 \otimes b - b \otimes 1$.

Proof. Consider the inclusion homomorphism

$$B \otimes f : B \otimes_A A \longrightarrow B \otimes_A B$$
$$b \otimes a = f(a)b \otimes 1 \longmapsto b \otimes f(a) = f(a)b \otimes 1.$$

The homomorphism $B \otimes f$ has a section, namely, the map $m : b \otimes b' \mapsto bb' \otimes 1$. In particular, $B \otimes f$ is injective, i.e., the sequence $0 \to B \otimes_A A \xrightarrow{B \otimes f} B \times_A B$ is exact. Moreover, the sequence

$$0 \to B \xrightarrow{B \otimes f} B \otimes_A B \xrightarrow{h} B \otimes_A B \otimes_A B$$

is exact, where $h = B \otimes (i_2 - i_1) = id_B \otimes (i_2 - i_1)$. To prove this, let

$$k : B \otimes_A (B \otimes_A B) \xrightarrow{(1,2)} B \otimes_A B \otimes B \xrightarrow{i_2 \cdot ((B \otimes f) \circ m)} (B \otimes_A B)$$
$$b \otimes b' \otimes b'' \longmapsto b' \otimes b \otimes b'' \longmapsto (1 \otimes b') \cdot [(B \otimes f) \circ m](b \otimes b'') = (bb'' \otimes b')$$

be a homomorphism. Then

$$k \circ h(b \otimes b') = k(b \otimes (1 \otimes b' - b' \otimes 1)) = k(b \otimes 1 \otimes b') - k(b \otimes b' \otimes 1) = bb' \otimes 1 - b \otimes b'.$$

Thus, if $b \otimes b' \in \mathrm{Ker}(h)$, we have $b \otimes b' = bb' \otimes 1 = B \otimes f(bb' \otimes 1) \in \mathrm{Im}(B \otimes f)$. On the other hand, it is clear that $h \circ (B \otimes f)(b \otimes 1) = b \otimes (1 \otimes 1 - 1 \otimes 1) = 0$.

Since f is faithfully flat, we have that the sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{i_2 - i_1} B \otimes_A B$$

is exact. □

Proposition 2.2.5. *Let X be a scheme, and let Z be an X-scheme. Then the presheaf*

$$h_Z : Et/X \longrightarrow \textbf{\textit{Set}}$$
$$U \longmapsto hom_{Et/X}(U, Z)$$

is a sheaf on the étale topology.

Proof. Let us prove the proposition in the case where $Z = Spec\ C$ is affine. It is easy to see that functor h_Z satisfies the sheaf criterion for open Zariski coverings. Therefore, it suffices to show that

$$hom_{Rings}(C, A) \longrightarrow hom_{Rings}(C, B) \rightrightarrows hom_{Rings}(C, B \otimes_A B)$$

is exact for any faithfully flat map $A \to B$. But this follows immediately from Proposition 2.2.4, since $hom_{Rings}(C, -)$ is left exact. So, by Proposition 2.2.2, h_Z is an étale sheaf.

Now, we prove the general case. Let Z be an arbitrary scheme, and let $\{Z_i\}_{i \in I}$ be an affine open cover. By sheaf criterion 2.2.2 we only have to check the sheaf condition for surjective étale morphism $h : V \to U$ of affine schemes U, V. Firstly, we show the injectivity of the map

$$hom_{Et/X}(U, Z) \to hom_{Et/X}(V, Z).$$

Let $f, g : U \to Z$ be two morphisms such that the composites $V \to U \to Z$ coincide. Since h is surjective, we know that f, g agree as functions of sets. Now, let $U_i := f^{-1}(Z_i) = g^{-1}(Z_i)$ and $V_i := h^{-1}(U_i)$. Since Z_i is affine, h_{Z_i} is a sheaf. In particular, we have that the following map is injective

$$hom_{Et/X}(U_i, Z_i) \to hom_{Et/X}(V_i, Z_i).$$

This means that $f|_{U_i} = g|_{U_i}$ for each $i \in I$, which implies that $f = g$.

Next, we show existence of gluing. Let $g \in hom_{Et/X}(V, Z)$ be an element such that both morphisms

$$hom_{Et/X}(V, Z) \underset{-\circ q}{\overset{-\circ p}{\rightrightarrows}} hom_{Et/X}(V \times_U V, Z)$$

agree, i.e., $g \circ q = g \circ p$, where $p,\ q$ are projection maps of fiber product $V \times_U V$. We need to show that there is a morphism $f \in hom_{Et/X}(U, Z)$ such that $g = f \circ h$. Define $V_i := g^{-1}(Z_i)$ and $U_i := h(V_i)$. We know that V_i is open in V, and since h is étale (and hence open), it follows that U_i is also open, implying that they are schemes. We know that the composites

$$V_i \times_{U_i} V_i \underset{q_i}{\overset{p_i}{\rightrightarrows}} V_i \xrightarrow{g|_{V_i}} Z_i$$

agree. Since Z_i is affine, the $g|_{V_i}$ factor uniquely through $f_i : U_i \to Z_i$. Now, we demonstrate that

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

Let $V_{i,j} := V_i \times_U V_j$, and let $U_{ij} := \mathrm{Im}(h_i|_{V_{i,j}} : V_{i,j} \xrightarrow{p_i} V_i \hookrightarrow V \xrightarrow{h} U)$. The morphism $h_i|_{V_{i,j}}$ is a composite of étale morphisms, so it is also étale. Thus, $U_{i,j}$ is an open subset. Since $h_i|_{V_{i,j}}(V_{i,j}) = U_{i,j}$ is open, the morphism of schemes

$$V_{i,j} \to V_i \hookrightarrow V \xrightarrow{h} U$$

factor through

$$V_{i,j} \xrightarrow{h_{i,j}} U_{i,j} \hookrightarrow U.$$

Since $U_{i,j}$ is an open subset, the morphism $U_{i,j} \hookrightarrow U$ is étale. Therefore, by 4) of Proposition 2.1.15, $h_{i,j}$ is an étale morphism. Since $h_{i,j}$ is surjective and $g_i|_{V_{i,j}}(V_{i,j}) \subseteq Z_i$, the morphism $g_i|_{V_{i,j}}$ factors uniquely through $f_{i,j} : U_{i,j} \to Z_i$, hence, by uniqueness $f_i|_{U_{i,j}} = f_{i,j}$ and similarly with i, j reversed. Since $g \circ p = g \circ q$, we have $g_i|_{V_{i,j}} = g_j|_{V_{i,j}}$, thus, by uniqueness, $f_{i,j} = f_{j,i}$. From the properties of fiber product, we have

$$U_i \cap U_j = h(V_i) \cap h(V_i) \subseteq U_{i,j}.$$

Thus, in particular, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. By Lemma 1.1.25, the f_i glue together to a morphism $f : U \to X$ such that $g = f \circ h$. This shows that

$$hom_{Et/X}(U, Z) \longrightarrow hom_{Et/X}(V, Z) \rightrightarrows hom_{Et/X}(V \times_U V, Z)$$

is exact. ◻

So, we can identify schemes with sheaves on the étale topology.

Remark 2.2.6. *If Z has a group structure, then h_Z is a sheaf of groups.*

Example 2.2.7. *1. Let $\mathbb{G}_a = Spec\ \mathbb{Z}[x]$ be an affine scheme (affine line). Note that*

$$hom_{Schemes}(U, \mathbb{G}_a) = hom_{Rings}(\mathbb{Z}[x], \mathcal{O}_U(U)).$$

Any ring map from $\mathbb{Z}[x]$ to another ring is completely determined by where x maps to, which can be any element of $\mathcal{O}_U(U)$. So the maps in $hom_{Schemes}(U, \mathbb{G}_a)$ correspond to elements of $\mathcal{O}_U(U)$

regarded as an abelian group. By the previous proposition,

$$\mathbb{G}_{a,X} := \mathbb{G}_a \times_{Spec\ \mathbb{Z}[x]} X : Et/X \to \textbf{\textit{Ab}}$$

is an étale sheaf on X_{et}, and for any étale X-scheme U,

$$\begin{aligned}\mathbb{G}_{a,X}(U) &= hom_{Et/X}(U, \mathbb{G}_{a,X}) \\ &= hom_{Schemes}(U, \mathbb{G}_a) \\ &= hom_{Rings}(\mathbb{Z}[t], \Gamma(U, \mathcal{O}_U)) \\ &= \Gamma(U, \mathcal{O}_U),\end{aligned}$$

where the second equality is guaranteed by universal property of fiber product, and $\Gamma(U, \mathcal{O}_U)$ is regarded as an abelian group.

2. *Let $\mathbb{G}_m = Spec\ \mathbb{Z}[x, x^{-1}]$ be an affine scheme (affine line with the origin omitted). Any ring map from $\mathbb{Z}[x, x^{-1}]$ to another ring R is uniquely determined by the image of x. Since x is invertible in $\mathbb{Z}[x, x^{-1}]$, the image under the homomorphism must be an invertible element in R. So, by the same reasoning, we have $\mathbb{G}_m(U) = \mathcal{O}_U(U)^{\times}$ for any scheme U. Furthermore, $\mathbb{G}_{m,X} := \mathbb{G}_m \times_{Spec\ \mathbb{Z}[x]} X$ is an étale sheaf on X_{et} and for any étale X-scheme U, we have $\mathbb{G}_{m,X}(U) = \Gamma(U, \mathcal{O}_U^{\times})$.*

3. *Let $\mu_n = Spec\ \mathbb{Z}[x]/(x^n - 1)$. Any ring map from $\mathbb{Z}[x]/(x^n - 1)$ to another ring R is uniquely determined by the image of $\overline{x} = x + (x^n - 1)$. Since $\overline{x}^n = 1$ in $\mathbb{Z}[x]/(x^n - 1)$, the image under the homomorphism must be an element of order n in R. So,*

$$\mu_n(U) := \{x \in \Gamma(U, \mathcal{O}_U) : x^n = 1\}$$

for any scheme U. Furthermore, $\mu_{n,X} = \mu_n \times_{Spec\ \mathbb{Z}} X$ is an étale sheaf on X_{et} and for any étale X-scheme U, we have

$$\mu_{n,X}(U) = \{x \in \Gamma(U, \mathcal{O}_U) : x^n = 1\}.$$

Remark 2.2.8. *Let X be a scheme. For any natural number n, we can define a sheaf morphism*

$$\begin{aligned}n_U : \mathbb{G}_{m,X}(U) &\longrightarrow \mathbb{G}_{m,X}(U) \\ s &\longmapsto s^n.\end{aligned}$$

It is easy to see that $\mu_{n,X}$ is a kernel of the map $\mathbb{G}_{m,X} \xrightarrow{n} \mathbb{G}_{m,X}$, so we have the following exact sequence of *abelian sheaves on* X_{et}

$$0 \to \mu_{n,X} \to \mathbb{G}_{m,X} \xrightarrow{n} \mathbb{G}_{m,X}.$$

The map $\mathbb{G}_{m,X} \xrightarrow{n} \mathbb{G}_{m,X}$ is surjective under suitable assumptions on n.

Theorem 2.2.9. *Let X be a scheme. Let n be invertible on X, i.e., n is invertible in $\mathcal{O}_X(X)$. Then there is an exact sequence*

$$0 \to \mu_{n,X} \to \mathbb{G}_{m,X} \xrightarrow{n} \mathbb{G}_{m,X} \to 0$$

*of morphisms of abelian sheaves on X_{et}. This sequence is called **the Kummer sequence on** X.*

Proof. We only have to show that the map $\mathbb{G}_{m,X} \xrightarrow{n} \mathbb{G}_{m,X}$ is surjective. For an étale X-scheme U, and $s \in \mathbb{G}_{m,X}(U) = \Gamma(U, \mathcal{O}_U)^{\times}$. If we can find a covering $\{U_i \to U\}$ of U in X_{et} such that the map $s_i \in \mathbb{G}_{m,X}(U_i) = \Gamma(U_i, \mathcal{O}_{U_i})^{\times}$, induced by s, are n-th power in $\Gamma(U_i, \mathcal{O}_{U_i})^{\times}$. Then, from the definition of the abelian sheaf, s is also an n-th power in $\Gamma(U_i, \mathcal{O}_{U_i})^{\times}$. Since any scheme has an open affine covering, we can assume that U is an affine scheme.

But this is a consequence of the following observation: if A is a ring with n invertible in A, and $s \in A^{\times}$, then the A-algebra $B := A[t]/(t^n - s)$ is free of rank n, in particular the inclusion map $i : A \hookrightarrow A[t]/(t^n - s)$ is faithfully flat, i.e., $Spec(i) : Spec\ B \to Spec\ A$ is surjective. Since $\frac{d}{dt}(t^n - s) = nt^{n-1}$ has an inverse $\frac{t}{ns}$ in $A[t]/(t^n - s)$, the map i is a standard étale homomorphism, in particular étale. Moreover, we have $\bar{t}^n = s$ where $\bar{t}$ is the image of t in $A[t]/(t^n - s)$. □

Example 2.2.10. *Let A be a discrete abelian group. We denote by A_X or simply $\underline{A}$ the sheaf associated to the presheaf $U \mapsto A$ for étale X-schemes U. A_X is called the **constant sheaf** with value in A. This sheaf is representable, as demonstrated on page 99 of [32]:*

$$A_X(U) = hom_{Et/X}(U, \coprod_A X).$$

2.3 Direct and inverse image

Next, we introduce the direct image and inverse image functors, which are specific geometric morphisms in the étale topology. These functors serve as tools for studying the relationship between the categories of sheaves on two different étale sites. These functors are analogous to the direct image and inverse image functors for sheaves on topological spaces.

Definition 2.3.1. *Let $\pi : Y \to X$ be a morphism of schemes, and let P be a presheaf on Y_{et}. The **direct image functor** $\pi_* : Psh(Y_{et}) \to Psh(X_{et})$ is given by*

$$\pi_* P(U) = P(U \times_X Y),$$

where $U \to X$ is an étale morphism. Since $U \times_X Y \to Y$ is étale (because it is a base change of étale morphism), π_* is well-defined.

Lemma 2.3.2. *If F is a sheaf, then also is $\pi_* F$.*

Proof. For an étale morphism $U \to X$, let U_Y denote the scheme $U \times_X Y$. Then $U \mapsto U_Y$ is a functor taking étale maps to étale maps, surjective families of maps to surjective families, and fiber products over X to fiber products over Y.

Let $\{U_i \to U\}_{i \in I}$ be an étale covering in Et/X of . Then $\{(U_i)_Y \to U_Y\}$ is an étale covering in Et/Y. And so

$$F(U_Y) \longrightarrow \prod_{i \in I} F((U_i)_Y) \rightrightarrows \prod_{i,j \in I} F((U_i)_Y \times_Y (U_j)_Y)$$

is exact. But this is equal to the sequence

$$\pi_* F(U) \longrightarrow \prod_{i \in I} (\pi_* F)(U_i) \rightrightarrows \prod_{i,j \in I} (\pi_* F)(U_i \times_X U_j)$$

which is also exact, as required.. □

So the restriction of π_*

$$\pi_* : Sh(Y_{et}) \to Sh(X_{et})$$

is well defined.

Proposition 2.3.3. *If the morphism π is finite, then the functor π_* is exact.*

Proof. See Proposition 8.3 and 8.4 of [21]. □

Let $\pi : Y \to X$ be a morphism of schemes. Now, we define a left adjoint for the functor π_*. Let P be a presheaf on X_{et}. For $V \to Y$ étale, we define

$$P^\dagger(V) := \varinjlim P(U)$$

where the injective limit runs over the commutative diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

with $U \to X$ étale. It is easy to see that $P^\dagger$ is a preshaef. And for any presheaf Q on Y, there are natural bijections between the following:

- morphism $P^\dagger \to Q$;
- families of functions $P(U) \to Q(V)$, indexed by commutative diagrams as described above, and compatible with restriction maps;
- morphism $P \to \pi_* Q$.

Hence,

$$hom_{Y_{et}}(P^\dagger, Q) \cong hom_{X_{et}}(P, \pi_* Q), \tag{2.1}$$

functorially in P and Q.

In general, $P^\dagger$ is not necessarily a sheaf, even if P is. Therefore, we define:

Definition 2.3.4. *Let $\pi : Y \to X$ be a morphism of schemes, and let P be a presheaf on Y_{et}. The **inverse image functor** $\pi^* : Sh(X_{et}) \to Sh(Y_{et})$ is given by*

$$\pi^* P = a(P^\dagger),$$

where $a(P^\dagger)$ is the étale sheaf associated with $P^\dagger$.

Proposition 2.3.5. *The direct image functor $\pi_* : Sh(Y_{et}) \to Sh(X_{et})$ is a right adjoint to π^*.*

Proof. The proof follows from the isomorphisms

$$hom_{Y_{et}}(\pi^* F, G) \cong hom_{Y_{et}}(F^\dagger, G) \cong hom_{X_{et}}(F, \pi_* G).$$

The first isomorphism uses the fact that sheafification is the left adjoint to the inclusion, and the second follows from the isomorphism 2.1. □

Proposition 2.3.6. *Let $\pi : Y \to X$ be a morphism of schemes. The inverse image functor π^* is exact.*

Proof. See Remark 8.9 [21] or 1.4.2, Chapter II of [32]. □

By Lemma 1.2.35, the direct image functor preserves injective objects.

Proposition 2.3.7. *Let $\pi : U \to X$ be an étale morphism of schemes. The inverse image functor π^* has an exact left adjoint $j_!$. In particular, π^* preserves injective objects.*

Proof. See Remark 8.16 of [21]. □

These properties are crucial for exploring the relationship between étale cohomology on two different étale sites.

Chapter 3

Real Algebra and Real Spectrum

In this chapter, we present some basic facts and concepts from real algebra and real algebraic geometry. The main references are [3], [2], [1], and [19]. In the first section, we introduce the positive cone, an algebraic description of the ”positive set”, and the notion of a real closed field, which serves as an analogue to algebraically closed fields in real algebra and real algebraic geometry.

In the second section, we introduce real ideals and prove the Real Nullstellensatz, which establishes a correspondence between real points and orderings in the coordinate rings.

In the third section, we present real closed valuation rings, convexity, and Archimedean property. We will also prove that a ring is a real closed valuation ring if and only if it is a convex subring of a real closed field.

In the final section, we introduce another key concept of the thesis: the real spectrum. We will explain why the real spectrum is useful in real algebraic geometry and provide a comparison between the Zariski spectrum and the real spectrum. Moreover, we will prove that the real spectrum of a real valuation ring is homeomorphic to its Zariski spectrum.

3.1 Cone and real field

In this thesis, an ordering of a field refers to a linear ordering that is compatible with both addition and multiplication.

Definition 3.1.1. *An* ***(compatible) ordering*** *of a field K is a total order relation $\leq$ with additional axioms:*
For any $x, y, z \in K$

i - $x \leq y \implies x + z \leq y + z$*;*

ii - $x \geq 0, y \geq 0 \implies xy \geq 0.$

An ordered field *is a field K equipped with an ordering* $\leq$*, denoted* $(K, \leq)$*.*

Remark 3.1.2. *The last axiom can be replaced by* $(z \geq 0, x \geq y \implies xz \geq yz)$*.*

There are several ways to study the ordering of a field; one of them is through the positive set of the field.

Definition 3.1.3 (Cone)**.** *A* ***cone*** *in a field K is a subset* $P \subseteq K$ *with induced operations satisfying the following properties:*

i - $P + P \subseteq P$*;*

ii - $P \cdot P \subseteq P$*;*

iii - $\sum K^2 \subset P$*.*

where $\sum K^2$ *is the set of sums of squares of K.*
We will say that a cone is ***proper*** *if* $-1 \notin P$*;*
We will say that a cone is ***positive*** *if it is proper and* $P \cup -P = K$*.*

Remark 3.1.4. *Note that if the cone P is proper, then* $P \cap -P = 0$*. Furthermore, the set of sums of squares forms a cone, and it is contained in every other cone.*

Proposition 3.1.5. $\sum K^2 \setminus \{0\}$ *is a multiplicative group.*

Proof. $1 = 1^2 \in \sum K^2 \setminus \{0\}$.

Now, let $x, y \in \sum K^2 \setminus \{0\}$ be two elements such that $x = x_1^2 + ... + x_n^2$ and $y = y_1^2 + ... + y_m^2$. We have that

$$xy = (x_1^2 + ... + x_n^2)(y_1^2 + ... + y_m^2) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i^2 y_j^2$$

and

$$x^{-1} = \frac{x}{x^2} = \left(\frac{x_1}{x}\right)^2 + \left(\frac{x_2}{x}\right)^2 + ... + \left(\frac{x_n}{x}\right)^2.$$

Therefore, $\sum K^2 \setminus \{0\}$ is a group. □

Now, we prove that in a field, the ordering and the positive cone are equivalent notions.

Proposition 3.1.6. *If K is a field and* $P \subset K$ *is a positive cone, then the relation* $\geq$ *(is sometimes also denoted by* $\geq_P$*) defined by*

$$x \geq y \iff x - y \in P$$

is an ordering.

Proof. Let's check the axioms, let $a, b, c \in K$

i - $a - a = 0 \in P$ is equivalent to $a \leq a$

ii - $(a - b \in P, b - c \in P \implies a - c \in P)$ is equivalent to $(a \geq b, b \geq c \implies a \geq c)$

iii - $(a - b \in P, b - a \in P \implies a = b)$ is equivalent to $(a \leq b, b \leq a \implies a = b)$

iv - $a - b \in K \iff a - b \in P$ or $a - b \in -P$ is equivalent to $a \leq b$ or $b \leq a$

v - $a - b = a + c - (b + c) \in P$ is equivalent to $(a \leq b \implies a + c \leq b + c)$

vi - $(a \in P, b \in P \implies ab \in P)$ is equivalent to $a, b \geq 0 \implies ab \geq 0$

□

Proposition 3.1.7. *Let $(K, \leq)$ be an ordered field, the subset $P := \{x \in K : x \geq 0\}$ is a positive cone of K.*

Proof. Let $a, b \in P, c \in K$, then we have:

i - $a + b \geq 0$ is equivalent to $a + b \in P$

ii - $ab \geq 0$ is equivalent to $ab \in P$.

iii - If $c \geq 0$, then $c^2 \geq 0$; if $c < 0$, then $-c > 0$. Therefore $c^2 = (-c)^2 \geq 0$, So we have $c^2 \in P$. From the previous items, $\sum K^2 \subset P$.

iv - Suppose that $-1 \in P$. Then, we have $0 = -1 + 1 \geq 0 + 1 = 1$. Since $0 \neq 1$ we have $0 > 1$, which implies that $1 \notin P$, a contradiction, as $1 = 1^2 \in P$ by the previous item.

v - By the definition of linear ordering, we have either $c \leq 0$, or $c \geq 0$, therefore $P \cup -P = K$.

□

So, there is a bijective correspondence between positive cones and compatible orderings. For this reason, studying an ordering of field is equivalent to studying a positive cone of a field. Since a cone has an algebraic structure, so it gains the advantage in some situations compared to the order relation.

Thus, we sometimes use the following definition of an ordered field.

Definition 3.1.8. *An **ordered field** is a field together with a positive cone, in other words, an ordered field is a pair (K, P), where K is a field and $P \subset K$ is a positive cone.*

Notation 3.1.9. *Let (K, P) be an ordered field, and let a, b be two elements of K such that $a <_P b$. We will denote the open interval between a and b in P (i.e., the set $\{x \in K : a <_P x <_P b\}$) by $(a, b)_P$. similarly, we denote the closed interval between a and b in P by $[a, b]_P$.*

Definition 3.1.10 (Real Field)**.** *A **(formally) real field** is a field that has an ordering, or equivalently, has a positive cone.*

The concepts of a "formally real field" and an "ordered field" are not the same. A "formally real field" is a field that has an ordering. In contrast, an "ordered field" is a field that is equipped with a total ordering that is compatible with the field operations.

Lemma 3.1.11. *Let $P \in K$ be a proper cone.*

i - If $-a \notin P$, then $P[a] = \{x + ay : x, y \in P\}$ is a proper cone of K.

ii - The cone P is contained in a positive cone of K

Proof. i - It is clear that $P \subset P[a] = \{x + ay : x, y \in P\}$ is a cone, we need to prove that $-1 \notin P[a]$: Suppose by absurdity, $x + ay = -1$ for some $x, y \in P$, if $y = 0$, we have $x = -1 \in P$, contradicts hypothesis, if $y \neq 0$, then y admits a inverse, therefore

$$x + ay = -1 \iff -ay = x + 1 \iff -a = (1/y)^2 y(x + 1) \in P,$$

contradicts hypothesis again. Thus we conclude that P is a proper cone.

ii - Using Zorn's lemma, there exists a maximal proper cone Q that contains P, we need to prove that $Q \cup -Q = K$: Let $-a \notin Q$, (it exists because $-1 \notin Q$,) thus, by the previous item, we have $Q[a]$ is a proper cone. Since Q is maximal, $Q = Q[a]$ and therefore $a \in Q[a] = Q$, as desired.

□

This lemma tells us that a positive cone is a maximal proper cone and if a field admits a proper cone, then it admits a positive cone, that is, it admits an ordering. Since the set of sums of squares is a cone contained in every other cone, we have the following theorem:

Theorem 3.1.12. *A field R is real if and only if $-1 \notin \sum R^2$.*

We had probably heard that the field of complex numbers has no order, with this, we can easily conclude that a complex number does not have an ordering compatible with addition and multiplication.

Proposition 3.1.13. *Let F be a field with $char(F) = 0$ and P be a proper cone of F. Then P is the intersection of all positive cones of F containing P.*

Proof. The cone P is necessarily contained in this intersection. If $a \notin P$, then $P[-a]$ is a proper cone by the Lemma 3.1.11 (i). Moreover the Lemma 3.1.11 (ii) ensures that there exists a positive cone containing $P[-a]$ but not a. □

Corollary 3.1.14. *If F is a real field, then $\sum F^2$ is the intersection of all positive cones of F.*

Definition 3.1.15 (Real Closed Field)**.** *Let R be a real field, R is said to be* ***real closed*** *if has no nontrivial algebraic extension that can be ordered.*

We have some equivalence of real closed field.

Theorem 3.1.16. *Let R be a real field, the following properties are equivalent:*

i - *R is real closed.*

ii - *$R^{(2)} = \{x^2 : x \in R\}$ is a unique positive cone of R, and every polynomial of $R[X]$, of odd degree, has a root in R.*

iii - *$R(i) = R[X]/ < X^2 + 1 >$ is an algebraically closed field.*

iv - *Let $a, b \in R, p \in R[X]$. If $p(a)p(b) < 0$, then p has a root $x \in R$ such that $a < x < b$.*

Proof. $i) \implies ii)$ Let $a \in R$, if $\sqrt{a} \notin R$ then $R' = R[X]/ < X^2 - a >$ is a nontrivial algebraic extension of R. Since R is a real closed field, R' is not a real field, therefore $-1 \in \sum(R')^2$, that is,

$$-1 = \sum_{i=1}^{n}(x_i + y_i\sqrt{a})^2 = \sum_{i=1}^{n} x_i^2 + a\sum_{i=1}^{n} y_i^2,$$

for some $x_i, y_i \in R$. Since R is a real field, we have $-1 \neq \sum_{i=1}^{n} x_i^2$, this implies that $0 \neq \sum_{i=1}^{n} y_i^2$. Since the set of non-zero sums of squares is a group (see Proposition 3.1.5),

$$-a = (\sum_{i=1}^{n} y_i^2)^{-1}(1 + \sum_{i=1}^{n} x_i^2) \in \sum R^2,$$

So, we can state that $R = \sum R^2 \cup -\sum R^2$, i.e., $\sum R^2$ is a positive cone (a maximal proper cone). Since $\sum R^2$ is contained in every positive cone, we conclude that $\sum R^2$ is the unique positive cone. Moreover, the equation above indicates that if $\sqrt{a} \notin R$ then a is negative, this is logically equivalent to every positive element admits a square root in R. Therefore $R^{(2)} = \sum R^2$ is a unique positive cone.

It remains to show that every polynomial in $R[X]$ of odd degree admits a root. Let's do it by induction: Let $\phi(n)$: If $f \in R[X]$, $deg(f) \leq 2n+1$ and $deg(f) \in (2\mathbb{N}+1)$, then there exists $x \in R$ such that $f(x) = 0$.

- $n = 0$: any polynomial $f = aX + b$ with $a \neq 0$ always vanishes in $\frac{-b}{a}$,
- $\phi(n)$ implies $\phi(n+1)$: Let $f \in R[X]$ be a polynomial of odd degree such that $deg(f) \leq 2n+3$. If $deg(f) < 2n+3$, then $deg(f) \leq 2n+1$, consequently f has a root. Suppose that $deg(f) = 2n+3$ and it does not have a root. Then f is irreducible: suppose not, $f = gh$, where $2n+3 > deg(g)$ and $deg(h) > 0$. Since $deg(f)$ is an odd and since

  $$deg(f) = deg(g) + deg(h),$$

  at least one between g *and* h has an odd degree, but this implies that f has a root by the hypothesis of induction, contradicts the hypothesis.

  Since f is irreducible, $R' = R[X]/fR[X]$ is an algebraic extension of R, and repeating the previous argument, we obtain

  $$-1 = (\sum_{i=1}^{n} h_i^2) + fg \in \sum (R')^2,$$

  where $deg(h_i) < 2n+3$.
  Since $deg(\sum_{i=1}^{n} h_i^2) < 4n+6$ is an even, $deg(g) < 2n+3$ is an odd, therefore by the induction hypothesis, g admits a root x. So, we have

  $$-1 = (\sum_{i=1}^{n} h_i^2(x)) + f(x)g(x) = \sum_{i=1}^{n} h_i^2(x) \in \sum R^2,$$

  contradicts the hypothesis. Thus, by principle of induction, every polynomial in $R[X]$ of odd degree admits a root.

$ii) \implies iii)$ For this implication, we need the following lemma:

Lemma 3.1.17. *Let K be a field, let $f \in K[X]$ be a polynomial of degree n, and let $x_1, ..., x_n$ be roots of f in an algebraically closed field C that contains K. If $Q(X_1, ...X_n) \in K[X_1, ...X_n]$ is symmetric, (i.e., $Q(X_1, ...X_n) = Q(X_{\sigma(1)}, ...X_{\sigma(n)})$ for any $\sigma \in S_n$), then $Q(x_1, ...x_n) \in K$ (See Proposition 2.24 of [2]).*

We want to prove that all even-degree polynomials have roots in R(i). We apply induction on $\phi(n)$: If $f \in R[X]$, $p := deg(f) = 2^n m$ and $m \in (2\mathbb{N}+1)$, then there exists $x \in R(i)$ such that $f(x) = 0$.

- $n = 0$: This case follows from the hypothesis.

- $\phi(n)$ implies $\phi(n+1)$: Let $x_1, ..., x_p$ be all the roots of f counting multiplicities in an algebraically closed field C, since $R(i)$ is an algebraic extension, $R(i) \subset C$. Now for each $h \in \mathbb{Z}$ we define

$$Q_h(X_1, ..., X_p, X) = \prod_{\lambda<\mu}(X - X_\lambda - X_\mu - hX_\lambda X_\mu).$$

This polynomial is symmetric in $X_1, ..., X_p$, therefore, by the lemma above, we have $Q_h(x_1, ..., x_p, X) \in R[X]$ with $deg(Q_h(x_1, ..., x_p, X)) = \binom{p}{2} = \frac{p(p-1)}{2}$. Note that $p-1$ is an odd, by the induction hypothesis, $Q_h(x_1, ..., x_p, X)$ has a root in $R(i)$, that is, there exist λ and μ such that $x_\lambda + x_\mu + hx_\lambda x_\mu \in R(i)$. Since $h \in \mathbb{Z}$ is arbitrary, $\mathbb{Z}$ is infinite and combinations of λ and μ are finite, there exist λ, μ, h *and* h' with $h \neq h'$ such that $x_\lambda + x_\mu + hx_\lambda x_\mu \in R(i)$ and $x_\lambda + x_\mu + h'x_\lambda x_\mu \in R(i)$. Thus, we have $x_\lambda + x_\mu \in R(i)$ and $x_\lambda x_\mu \in R(i)$. Note that x_λ and x_μ are roots of $X^2 - (x_\lambda + x_\mu)X + x_\lambda x_\mu \in R(i)[X]$. The discriminant of this polynomial is $(x_\lambda - x_\mu)^2 \geq 0$. From the hypothesis, the square root of the discriminant exists, hence the roots x_λ, x_μ of f are inside $R(i)$.

Therefore, every polynomial in $R[X]$ has a root in $R(i)$.

It remains to show that $f \in R(i)[X]$ also has a root in $R(i)$. In this case we have $f = g + hi$, where $g, h \in R[x]$. Since $f\overline{f} = (g + hi)(g - hi) = g^2 + h^2 \in R[x]$, there exists $x \in R(i)$ such that $f\overline{f}(x) = 0$, i.e., either $f(x) = 0$ or $\overline{f}(x) = 0$. If $f(x) = 0$, we have what we need. If $\overline{f}(x) = 0$ we have $f(\overline{x}) = 0$. Thus, f has a root in $R(i)$, as desired.

iii) $\implies$ *i)* Since $R(i)$ is algebraically closed and contains R, the only proper algebraic extension of R is $R(i)$ which is not real ($-1 = i^2 \in \sum R(i)^2$), it suffices to prove that R is real. Since there does not exist $i \in R$ such that $i^2 = -1$, if we can prove that all sums of squares are squares, we conclude the proof: By the hypothesis, for every $a, b \in R$, there exist $c, d \in R$, such that $a + ib = (c + id)^2$, multiplying both sides by the conjugate we obtain $a^2 + b^2 = (c^2 + d^2)^2$.

iii) $\implies$ *iv)*: Let $f \in R[X]$, and let $a, b \in R$ such that $f(a)f(b) < 0$. Since $R(i)[X]$ is algebraically closed, then f can be written as linear factors $f = (X - a_1 - b_1 i)....(X - a_n - b_n i)$. Since at a and b, f has opposite sign, then some factors have opposite sign at a and at b. Without loss of generality, suppose that $(a - a_1 - b_1 i) > 0$ and $(b - a_1 - b_1 i) < 0$. Since $R[i]$ is not a real field, we have $b_1 = 0$, therefore $a_1 \in R$ satisfies $b < a_1 < a$ and is a root of f.

iv) $\implies$ *ii)*: Since a monic polynomial of odd degree $f(x) \in R[X]$ tends to $+\infty$ (respectively $-\infty$) when x tends to $+\infty$ (respectively $-\infty$), there exist $a, b \in R$ such that $f(a)f(b) < 0$, by the hypothesis, f admits a root in R.

We need to prove that all positive numbers in R admit a square root: If $p \in R$ is a non-zero positive element (for 0 is trivial), we consider $f(X) = X^2 - p$. since $f(0) = -p < 0$ and $f(p+1) = p^2 + p + 1 > 0$, by the hypothesis, f admits a root in R as desired. ☐

Every real field has a real closed extension, and this extension is unique up to isomorphism.

Definition 3.1.18. *A **real closure** of a real field is an algebraic extension that is also a real closed field.*

Theorem 3.1.19. *Every ordered field F has a real closure. If R and R' are two real closures of F, then there is a unique F-isomorphism $\phi : R \to R'$.*

Proof. See 1.3.2 of [3]. ☐

Proposition 3.1.20. *Let F be an ordered field, let R be a real closure of F, and let R' be a real closed extension of F whose ordering extends that of F. Then there exists a unique F-homomorphism $\phi : R \to R'$. In particular, if R' is also a real closure of F, then the homomorphism ϕ is an isomorphism.*

Proof. This is the Proposition 1.3.4 of [3]. ☐

3.2 Real Nullstellensatz

First, we prove the Artin-Lang homomorphism theorem using a result from model theory. This theorem will then be applied to prove the Real Nullstellensatz.

Theorem 3.2.1 (Artin-Lang homomorphism theorem)**.** *Let R' and R be real closed fields such that $R \subset R'$, and let A be an R-algebra of finite type. If there exists a homomorphism of R-algebras $\phi : A \to R'$, then there exists a homomorphism of R-algebras $\psi : A \to R$.*

Proof. Since A is an Algebra of finite type, it can be represented by $R[X_1, ..., X_n]/I$ for some natural n, furthermore by Hilbert's basis theorem, I is finitely generated, that is, there exist $P_1, ..., P_m$ as generators of I. Let $\phi : R[X_1, ..., X_n]/I \to R'$ be a homomorphism, by definition of homomorphism,

$$0 = \phi(P_i) = \phi(P_i(X_1, ..., X_n)) = P_i(\phi(X_1), ...\phi(X_n))$$

for $i = 1, ..., m$. This means

$$R' \models (\exists X_1)...(\exists X_n)(P_1(X_1, ..., X_n) = ... = P_m(X_1, ..., X_n) = 0).$$

Since the theory of real closed fields is model complete (see 3.3.16 of [18]), we have

$$R \models (\exists X_1)...(\exists X_n)(P_1(X_1, ..., X_n) = ... = P_m(X_1, ..., X_n) = 0).$$

In other words, there exists $(a_1, ..., a_n) \in R^n$ such that $P_1(a_1, ..., a_n) = ... = P_m(a_1, ..., a_n) = 0$.

Finally, we can define an R-homomorphism $\overline{\psi} : R[X_1, ..., X_n] \to R$ with $\overline{\psi}(X_i) = a_i$. Since $(a_1, ..., a_n)$ is a solution of $P_1(X_1, ..., X_n) = ...P_m(X_1, ..., X_n) = 0$,

$$\overline{\psi}(P_1(X_1, ..., X_n)) = ... = \overline{\psi}(P_m(X_1, ..., X_n)) = 0.$$

This implies that $I \subset \text{Ker } \overline{\psi}$, therefore by homomorphism theorem, we have a homomorphism $\psi : R[X_1, ..., X_n]/I \to R$ such that the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{\overline{\psi}} & R' \\ {\scriptstyle \pi}\downarrow & \nearrow_{\psi} & \\ R/I. & & \end{array}$$

□

Definition 3.2.2. *Let A be a ring (commutative with unity). An ideal I of A is said to be **real ideal** if and only if, for any $a_1, ..., a_n \in A$, we have*

$$a_1^2 + ... + a_n^2 \in I \text{ implies that } a_i \in I \text{ for } i = 1, ..., n.$$

Hilbert's Nullstellensatz theorem is restricted to algebraically closed fields, however, it is amazing that we can establish an analogue of the Nullstellensatz for real closed fields using real ideal. Let us prove some results about real ideals firstly, and then we prove the Real Nullstellensatz.

Lemma 3.2.3. *Every real ideal I of a ring A is a radical ideal. Furthermore, if A is Noetherian, then every minimal prime ideal that contains I is real. In particular, if A is Noetherian and I is a real ideal of A, then there exist finite prime real ideals $\mathfrak{p}_1, ..., \mathfrak{p}_q$ such that $I = \bigcap_{i=1}^{q} \mathfrak{p}_i$*

Proof. If $a^n \in I$, then we have

- If n is an even, then, by definition of a real ideal, $a^{\frac{n}{2}} \in I$;
- If n is an odd, then since $a^{n+1} = a \cdot a^n \in I$, by definition of a real ideal, $a^{\frac{n+1}{2}} \in I$

In both cases, the power decreases, indicating that through iterative processing, we can conclude that $a \in I$. This demonstrates that I is a radical ideal.

Since I is a radical, then we have $\bigcap_{\mathfrak{p} \in Spec(I)} \mathfrak{p} = I$. Let $\mathfrak{p}$ be a minimal prime ideal of I. Since A is Noetherian, there are $\mathfrak{p}_1, ..., \mathfrak{p}_q \in Spec(I)$ minimal prime ideals of I such that $\bigcap_{i=1}^{n} \mathfrak{p}_i = I$. If

$q = 1$, we are done, since $\mathfrak{p}_1 = I$ is real. Suppose that $q > 1$, if $\mathfrak{p}_1$ is not real, then there exist $a_1, .., a_k \in A - \mathfrak{p}_1$ such as $a_1^2 + ... + a_k^2 \in \mathfrak{p}_1$. We choose $b_i \in \mathfrak{p}_i - \mathfrak{p}_1$ (this set is nonempty, since $\mathfrak{p}_1$ is minimal), for $i = 2, ..., q$, and define $b := \prod_{i=2}^{q} b_i$. Then $(a_1 b)^2 + ... + (a_k b)^2 \in \bigcap_{i=1}^{n} \mathfrak{p}_i = I$, since I is real, $a_1 b \in I \subseteq \mathfrak{p}_1$, contradicts that $a_1 b \notin \mathfrak{p}_1$ (since $a_1, b \notin \mathfrak{p}_1$). Since $\mathfrak{p}_1$ is arbitrary, we conclude that all minimal prime ideals containing I are real. □

Lemma 3.2.4. *Let A be a ring (commutative with unity), and let I be a prime ideal of A. I is real if and only if the field of fractions of A/I is real.*

Proof. It is easy to see that a field F is real if and only if for all $x_1, ..., x_n \in F$, $\sum_{i=1}^{n} x_i^2 = 0$ implies that $x_1 = ... = x_n = 0$. The proof follows directly from the definition of a real ideal and this equivalence. □

Lemma 3.2.5. *Let A be a ring (commutative with unity), I an ideal of A. Then,*

$$\sqrt[R]{I} = \{a \in A : \exists m \in \mathbb{N},\ \exists b_1, ..., b_n \in A\ \ a^{2m} + b_1^2 + ... + b_n^2 \in I\}$$

is the smallest real ideal of A containing I. The ideal $\sqrt[R]{I}$ is said to be the ***real radical*** *of I. Moreover, if A is Noetherian, then $\sqrt[R]{I}$ is equal to the intersection of all real prime ideals containing I (or I is proper A, in which case there is no prime ideal that contains I).*

Proof. Let's prove $\sqrt[R]{I}$ is a ideal first. The difficult part is to verify that $\sqrt[R]{I}$ is closed under addition. Suppose

$$a^{2m} + b_1^2 + ... + b_n^2 \in I \ \ \textit{and}\ \ (a')^{2m'} + (b_1')^2 + ... + (b_n')^2 \in I.$$

We can write

$$(a + a')^{2(m+m')} + (a - a')^{2(m+m')} = a^{2m} c + (a')^{2m'} c',$$

where c and c' are the sum of the squares of the elements of A. Then

$$c(a^{2m} + b_1^2 + ... + b_n^2) + c'((a')^{2m'} + (b_1')^2 + ... + (b_n')^2) \in I,$$

i.e.,

$$(a + a')^{2(m+m')} + (a - a')^{2(m+m')} + c(b_1^2 + ... + b_n^2) + c'((b_1')^2 + ... + (b_n')^2) \in I,$$

therefore, $a + a' \in \sqrt[R]{I}$.

It is straightforward to observe that $\sqrt[R]{I}$ is real. Here is some immediate facts about real radicals:

- If I, J are ideals with $I \subseteq J$, then $\sqrt[R]{I} \subseteq \sqrt[R]{J}$.

- Every real ideal is itself real radical.

Hence, for every real ideal J containing I, $\sqrt[R]{I} \subseteq \sqrt[R]{J} = J$, this demonstrates that $\sqrt[R]{I}$ is the smallest real ideal containing I.

By the Lemma 3.2.3, any minimal prime ideal of $\sqrt[R]{I}$ is a real prime ideal containing I. Since a real radical ideal is radical,

$$\sqrt[R]{I} \subseteq \bigcap_{\mathfrak{p} \text{ is real prime ideal containing } I} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \text{ is minimal prime ideal of } \sqrt[R]{I}} \mathfrak{p} = \sqrt[R]{I}$$

i.e., $\sqrt[R]{I}$ is the intersection of all real prime ideals containing I. □

For the last statement, the assumption that A is Noetherian can be omitted, see the Proposition 4.1.7 of [3].

Theorem 3.2.6. *Let R be a real closed field and I an ideal of $R[X_1, ..., X_n]$. Then $I = I(V(I))$ if and only if I is real.*

Proof. Assume that $I = I(V(I))$. If $P_1, ..., P_s$ are polynomials such that $P_1^2 + ... + P_s^2 \in I$, then $P_i(x) = 0$ for every $x \in V(I)$ and $i = 1, ..., s$. Hence, $P_i \in I$, for $i = 1, ..., s$. We prove that I is real.

We begin by proving the theorem for the case of a real prime ideal, and then generalize it to any real ideal. Assume that J is a real prime ideal. It is clear that $J \subseteq I(V(J))$, so we need to prove that for any $P \in R[X_1, ..., X_n] \setminus J$, $P \notin I(V(J))$. We denote the image of P in the residue ring $B := R[X_1, ..., X_n]/J$ by $\overline{P}$. We choose an ordering of the field of fractions of B, which is possible by Lemma 3.2.4, let R_1 be a real closure of this ordered field. Let $A := B_{\overline{P}}$ be a ring. It is clear that A is a finite type R-algebra that is contained in R_1 (there is an inclusion homomorphism), by the Artin-Lang homomorphism Theorem 3.2.1, there is a homomorphism of the R-algebra $\psi : A \to R$. We define $x = (\psi(\overline{X_1}), ..., \psi(\overline{X_n}))$. Then $Q(x) = \psi(\overline{Q}) = \psi(0) = 0$ for all $Q \in J$, therefore $x \in V(J)$. However, since $\overline{P}$ is invertible in A, $P(x) = \psi(\overline{P}) \neq 0$, this shows that $P \notin I(V(J))$. Hence, $J = I(V(J))$.

Now, let I be a arbitrary real ideal. By Lemma 3.2.3, there exist real prime ideals $\mathfrak{p}_1, ..., \mathfrak{p}_q \in Spec(I)$ such that $\bigcap_{i=1}^n \mathfrak{p}_i = I$. Thus, $f \in I(V(I))$ implies that $f(x) = 0$ for all $x \in V(I) = \bigcup_{i=1}^n V(\mathfrak{p}_i)$, i.e., $f \in \bigcap_{i=1}^n \mathfrak{p}_i = I$. This shows that $I \subseteq I(V(I))$, since $I(V(I)) \subseteq I$ is obvious, we conclude that $I = I(V(I))$. □

Corollary 3.2.7 (Nullstellensatz real)**.** *Let R be a real closed field and I be an ideal of $R[X_1, ..., X_n]$. Then $\sqrt[R]{I} = I(V(I))$.*

Proof. Use $\sqrt[R]{I}$ equal to the intersection of all real prime ideals containing I and apply the previous theorem. □

Corollary 3.2.8. *Let $Z \subseteq R^n$ be an irreducible algebraic set, and let $I \subseteq R[Z]$ be an ideal. Then,*

$$I_Z(V(I)) := \{P \in R[Z] : P(x) = 0 \ \ \forall x \in V(I)\} = \sqrt[R]{I}$$

.

Unlike algebraic sets defined by a non-constant polynomial f in an algebraically closed field, an algebraic set defined by a non-constant polynomial in a real closed field can be empty. For example, the maximal ideal $(X^2+1) \subseteq \mathbb{R}[X]$ corresponds to an empty algebraic set. The Real Nullstellensatz ensures that $\sqrt[R]{I} = I(V(\varnothing)) = R[X_1, ..., X_n]$, so a ideal I corresponds to a non-empty algebraic set in R^n if and only if I is contained in a real ideal. For this reason, a maximal ideal corresponds to a point if and only if it is real.

Theorem 3.2.9. *Let R be a real closed field, and let $A = R[X_1, ..., X_n]$. Then an ideal $\mathfrak{m}$ is a real maximal ideal of A if and only if $\mathfrak{m}$ is of the form $(X_1 - a_1, ..., X_n - a_n)$ for some $a_1, ..., a_n \in R$.*

Proof. Let $a = (a_1, ..., a_n) \in R^n$ be any element, and let $\mathfrak{m} = (X_1 - a_1, ..., X_n - a_n)$. The function $A \to R$, with $p \mapsto p(a)$ is trivially a homomorphism with kernel equal to $\mathfrak{m}$. Therefore, by the isomorphism theorem, we have $A/\mathfrak{m} \cong R$, so $\mathfrak{m}$ is maximal, and by the Lemma 3.2.4, $\mathfrak{m}$ is real.

Let $\mathfrak{m}$ be a maximal and real ideal. Since A is a finite type R-algebra, $A/\mathfrak{m}$ is a real algebraic extension of R. By the definition of real closed field, $R = A/\mathfrak{m}$. Therefore, there is a natural isomorphism

$$\phi : R \hookrightarrow A \xrightarrow{\pi} A/\mathfrak{m} = R.$$

Let $b_i := \pi(X_i)$ and $a_i := \phi^{-1}(b_i)$. Then for each i, we have

$$X_i - a_i \in \ker \pi = \mathfrak{m}.$$

And therefore,

$$(X_1 - a_1, ..., X_n - a_n) \subseteq \mathfrak{m}.$$

But $(X_1 - a_1, ..., X_n - a_n)$ is a maximal ideal, hence, we have the equality $(X_1 - a_1, ..., X_n - a_n) = \mathfrak{m}$. □

Hence, there is a bridge connecting the orderings and the real algebraic sets.

Corollary 3.2.10. *Let R be a real closed field and let $A = R[X_1, ..., X_n]$. The maps $V : \{ideals\ of\ A\} \to \{subsets\ of\ \mathbb{A}^n_R\}$, $I : \{subsets\ of\ \mathbb{A}^n_R\} \to \{ideals\ of\ A\}$ induce the bijective functions between*

- *$\{algebraic\ sets\ in\ \mathbb{A}^n_R\}$ and $\{real\ ideals\ of\ A\}$;*

- $\{irreducible\ sets\ in\ \mathbb{A}^n_R\}$ and $\{real\ prime\ ideals\ of\ A\}$;
- $\{points\ in\ \mathbb{A}^n_R\}$ and $\{real\ and\ maximal\ ideals\ of\ A\}$

Proof. The proof follows from Theorem 3.2.7 and Theorem 3.2.9. □

In particular, the ordering of a field is closely related to the real points of algebraic varieties. This connection allows the ordering of a field to be used in the study of Diophantine equations via the Hasse principle. Given a polynomial equation $f(x) = 0$ with rational coefficients, if it has a rational solution, then it also has a real solution and a solution in the p-adic numbers. A polynomial $f(x)$ is said to satisfy the Hasse principle if the reverse is also true. If $f(x)$ satisfies the Hasse principle, then an ordering on the coordinate ring associated with $f(x)$ can be seen as part of the "structure" of a rational point. By identifying corresponding "parts" of a rational point in the p-adic field, these local solutions can be combined to form a global solution.

3.3 Convex subring

In this section, we present real closed valuation rings, convexity, and Archimedean property. We will also prove that a ring is a real closed valuation ring if and only if it is a convex subring of a real closed field.

Definition 3.3.1. *A **real closed valuation ring** is a valuation ring whose residue field and whose field of fractions are both real closed.*

Definition 3.3.2. *Let (K, P) be an ordered field. A subset $M \subseteq K$ is said to be P-**convex** if*

$$a, b \in M,\ c \in K,\ a <_P c <_P b \implies c \in M.$$

The smallest P-convex set that contains a given set $M \subseteq K$ is called the P-convex hull of M in K.

Proposition 3.3.3. *If (K, P) is an ordered field, every P-convex subring of K is a valuation ring of K.*

Proof. Let $B \subseteq K$ be a P-convex subring. Since $1 \in B$, we have $[-1, 1]_P \subseteq B$. So if $a \in K^\times$ satisfies $|a| \leq_P 1$, then $a \in B$. If $|a| >_P 1$, then $a^{-1} \in B$. Hence B is a valuation ring. □

Proposition 3.3.4. *Let (K, P) be an ordered field and let A be a subring of K.*

1. *The P-convex hull of A in K is a subring of K. In particular, it is a valuation ring of K.*
2. *A is P-convex in K if and only if $[0, 1]_P \subseteq A$.*
3. *If A is P-convex in K, then so is every A-submodule of K.*

Proof. 1. Let B be a P-convex hull of A in K. Then

$$B = \bigcup_{a \in A} [-|a|, |a|]$$

which is clearly an additive subgroup of K. It remains to prove that B is closed by multiplication. If $x, y \in B$, then $x \in [-a, a]_P$ and $y \in [-b, b]_P$ for some $a, b \in A$. Since $-ab \leq_P xy \leq_P ab$, we have $xy \in B$ as desired.

2. If A is P-convex in K, it's obvious that the interval $[0,1]_P$ is a subset of A. Assume $[0,1]_P \subseteq A$. Then, for every $a, b \in A$ and $c \in K$ such that $a <_P c <_P b$, we have $a + t(b-a) \in A$ for all $t \in [0,1]_P$. In particular, $c = a + \frac{c-a}{b-a}(b-a)$.

3. Let M be an A-submodule of K. If $x \in M$ such that $x \geq_P 0$, then

$$[-x, x]_P = \{ax : a \in K, a \in [-1,1]_P\}$$

is contained in $Ax \subseteq M$, which shows that M is P-convex.

□

Corollary 3.3.5. *Let (K, P) be an ordered field and let B be a valuation ring of K. The following properties are equivalent:*

1. *B is P-convex in K.*
2. *Every prime ideal $\mathfrak{p}$ is P-convex in B.*
3. *If $\mathfrak{p}$ is a prime ideal, then, for every $a \in \mathfrak{p}$, one has $-1 <_P a <_P 1$.*
4. *For every $a \in m_B$, one has $-1 <_P a <_P 1$.*

Proof. 1) $\implies$ 2) : holds by the previous proposition.
2) $\implies$ 3) : if $1 <_P a$, then $1 \in \mathfrak{p}$, a contradiction.
3) $\implies$ 4) : by the hypothesis, such property holds for the maximal ideal m_B.
4) $\implies$ 1) It suffices to show $[0,1]_P \subseteq B$. Let $x \in [0,1]_P$, if $x \notin B$, we have $x^{-1} \in B$, hence, $x^{-1} \in m_B$. But $x^{-1} >_P 1$, this is a contradiction. □

Let (K, P) be an ordered field. And let B be a P-convex subring of K. It's intuitive that the ordering P induces an ordering on the quotient field of $\kappa(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of B. We will now discuss how this ordering can be constructed. Let $\mathfrak{p}$ be a prime ideal of B, and let

$\kappa(\mathfrak{p}) = Frac(B/\mathfrak{p})$ be its quotient field. We will denote the image of a in $\kappa(\mathfrak{p})$ by $\overline{a}$. The subset

$$\overline{P_{\mathfrak{p}}} := \left\{ \frac{\overline{a}}{\overline{b}} : a, b \in B,\ a, b \geq_P 0,\ b \notin \mathfrak{p} \right\}$$

of $\kappa(\mathfrak{p})$ forms a positive cone in $\kappa(\mathfrak{p})$. It is clear that $\overline{P_{\mathfrak{p}}} \cdot \overline{P_{\mathfrak{p}}} \subseteq \overline{P_{\mathfrak{p}}}$, $\overline{P_{\mathfrak{p}}} + \overline{P_{\mathfrak{p}}} \subseteq \overline{P_{\mathfrak{p}}}$, and $\overline{P_{\mathfrak{p}}} \cup -\overline{P_{\mathfrak{p}}} = \kappa(\mathfrak{p})$. Now, assume that $-1 \in \overline{P_{\mathfrak{p}}}$, then there exist $a, b \in B$ with $a, b \geq_P 0$ and $b \notin \mathfrak{p}$ for which $-1 = \frac{\overline{a}}{\overline{b}}$. This implies that $a + b \in \mathfrak{p}$. However, since $0 \leq b \leq_P a + b$, by the previous proposition, $\mathfrak{p}$ is P-convex, we must have $b \in \mathfrak{p}$, which is a contradiction. Therefore, we conclude that:

Corollary 3.3.6. *If B is a convex subring of an ordered field (K, P), then B is a valuation ring of K, and the quotient field $\kappa(\mathfrak{p})$ is real for every $\mathfrak{p} \in Spec\ B$.*

If K is a real closed field, we have stronger results.

Proposition 3.3.7. *Let R be a real closed field. For every convex subring B of R and $\mathfrak{p} \in Spec\ B$. The quotient field $\kappa(\mathfrak{p})$ of B is real closed.*

Proof. Let $f \in B[t]$ be a monic polynomial of odd degree. Since R is real closed, f has a root $x \in R$. Because of B is a valuation ring, B is normal, therefore $x \in B$. Since every polynomial $\overline{f} \in \kappa(\mathfrak{p})[t]$ has a representative $f \in B[t]$, every polynomial in $\kappa(\mathfrak{p})[t]$ of odd degree has a root. Moreover, the set of squares in $\kappa(\mathfrak{p})$ is a positive cone $\overline{R^{(2)}_{\mathfrak{p}}}$ corresponding to the $R^{(2)}$ of $\kappa(\mathfrak{p})$. So $\kappa(\mathfrak{p})$ is real closed. □

In particular we have

Corollary 3.3.8. *A ring is a real closed valuation ring if and only if it is a convex subring of a real closed field.*

Proof. The previous proposition guarantees that every convex subring of a real closed field is a real closed valuation ring. Let B be a real closed valuation ring. Now we will prove that B is a convex subring of its field of fractions. Note that the unique positive cone of $\kappa = B/m_B$ is

$$\overline{P} := \{\overline{a^2} : a \in B\}$$

since κ is real closed. From the definition of positive cone, $-1 \notin \overline{P}$, hence $a^2 + 1 \notin m_B$ for every $a \in B$. In other words, $x <_P 1$ for every $x \in m_B$. Hence, by multiplying both sides by -1, we obtain $x > -1$ for every $x \in m_B$. From the Corollary 3.3.5, B is a convex subring. □

Definition 3.3.9. *Let (K, P) be an ordered field and A be a subring of K. We say that K is (relatively)* ***archimedean*** *over A with respect to P if, for every $b \in K$, there is $a \in A$ such that $b \leq_P a$. It is equivalent to say that K is the P-convex hull of A in K.*

Proposition 3.3.10. *If (K, P) is an ordered field, then the real closure R of K is archimedean over K.*

Proof. Let $x \in R$ be a non zero element, from the definition of algebraic extension, there exist $a_0, ..., a_n \in K$ and an irreducible polynomial $f(X) = a_0 + ... + a_n X \in K[X]$ such that $f(x) = 0$. If $|x| > y$ for every $y \in K$, we have $|r||x| > |r|\frac{y}{|r|} = y$ for every $r \in K^\times$ and $y \in K$. So,

$$0 = |a_0 + ... + a_n x^n| \geq |a_1 x ... + a_n x^n| - |a_0| \geq |a_1 + a_2 x + ... + a_n x^{n-1}||x| - |a_0|$$

Since $f(X)$ is irreducible, $|a_1 + a_2 x + ... + a_n x^{n-1}| \neq 0$, hence $0 \geq y - |a_0|$ for every $y \in K$, this is a contradiction. □

3.4 Real spectrum

The Zariski spectrum of a ring A is the space of all prime ideals with a (closed) topology given by the "algebraic subset". The idea of real spectrum of a ring A is analogous, it's the space of all ordering of the ring with a (open) topology given by the "open semi-algebraic subset".

First, we characterize the ordering of the ring by cone.

Definition 3.4.1. *Let A be a commutative ring . A **cone** P of A is a subset of A satisfying:*

1. $P + P \subseteq P$;
2. $P \cdot P \subseteq P$;
3. $\sum K^2 \subseteq P$.

where $\sum K^2$ is the set of all the sums of the squares of the elements of K.

*The cone P is said to be **proper** if $-1 \notin P$.*

In general, we cannot define an ordering even if a proper cone exists, because we cannot assign a sign to certain elements, such as zero divisors. Therefore, we can only define an ordering for a domain, specifically in the quotient ring $A/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal. Since every ordering of a domain can be extended to its quotient field (field of fractions), and every ordering of the field induces an ordering on the domain, we can focus on the ordering of the quotient field. This motivates us to define the prime cone, rather than the positive cone.

Definition 3.4.2. *Let A be a commutative ring. A **prime cone** P of A is a proper cone of A satisfying:*

$$ab \in P \implies a \in P \text{ or } -b \in P.$$

Proposition 3.4.3. *Let P be a prime cone of A, then:*

1. $P \cup -P = A$.

2. $P \cap -P$ *is a prime ideal of* A*, called the* ***support*** *of* P *and denoted by* $supp(P)$.

Proof. See the Proposition 4.3.2 of [3] □

Proposition 3.4.4. *Let* A *be a commutative ring. A subset* $P \subseteq A$ *is a prime cone if and only if there exists an ordered field* $(F, \leq)$ *and a homomorphism* $\phi : A \to F$*, such that* $P = \{a \in A : \phi(a) \geq 0\}$.

Proof. See the Proposition 4.3.4 of [3]. □

Proposition 3.4.5. *Let* A *be a commutative ring. A subset* $P \subseteq A$ *is a prime cone if and only if the image of* P *under the canonical homomorphism* $A \to \kappa(supp(P))$

$$\overline{P} = \left\{ \frac{\overline{a}}{\overline{b}} \in \kappa(supp(P)) : ab \in P \right\}$$

is the positive cone of an ordering of $\kappa(supp(P))$*. In particular,* $supp(P)$ *is a real prime ideal.*

Proof. See Proposition 4.3.4 and 4.3.5 of [3]. □

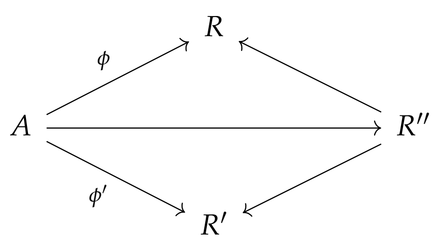
We have three equivalent definitions of points in the real spectrum.

Proposition 3.4.6. *Let* A *be a ring. The following data sets are equivalent:*

1. *a prime cone* α *of* A.

2. *a pair* $(\mathfrak{p}, \leq)$*, where* $\mathfrak{p}$ *is a prime ideal of* A *and* α *is an ordering of the quotient field* $\kappa(\mathfrak{p})$.

3. *An equivalent class of homomorphisms* $\phi : A \to R$ *with values in a real closed field, the equivalence relation is given by:* $\phi : A \to R$ *and* $\phi' : A \to R'$ *are equivalent if and only if there is a commutative diagram*



where R'' *is also a real closed field.*

In detail, one goes from 1) *to* 2) *by taking* $(\mathfrak{p}, \leq) = (supp(\alpha), \leq_\alpha)$*, from* 2) *to* 3) *by taking* $\phi : A \to \kappa(\mathfrak{p}) \to R$*, where* R *is the real closure of* $\kappa(\mathfrak{p})$ *for* $\leq$*, and from* 3) *to* 1) *by taking* $\alpha = \{a \in A : \phi(a) \leq 0\}$.

Notation 3.4.7. *Let* A *be a commutative ring. Let* $(\mathfrak{p}, \alpha)$ *be a pair where* $\mathfrak{p}$ *is a prime ideal of* A *and* α *is an ordering of the quotient field* $\kappa(\mathfrak{p})$*. The real closure of the ordered field* $(\kappa(\mathfrak{p}), \alpha)$ *is written by* $\kappa(\alpha)$.

For $a \in A$*, we writes*

1. $a(\alpha) \geq 0$ *(or* $a \geq_\alpha 0$*) if and only if the image of a in* $\kappa(\alpha)$ *is positive or equal to zero.*

2. $a(\alpha) > 0$ *(or* $a >_\alpha 0$*) if and only if the image of a in* $\kappa(\alpha)$ *is strictly positive.*

3. $a(\alpha) = 0$ *(or* $a =_\alpha 0$*) if and only if the image of a in* $\kappa(\alpha)$ *is equal to zero.*

Definition 3.4.8. *The* ***real spectrum*** *of* A*, denoted by* $Sper\ A$*, is the topological space whose points are the pairs* $(\mathfrak{p}, \alpha)$*, where* $\mathfrak{p}$ *is a prime ideal of* A *and* α *is an ordering of the quotient field* $\kappa(\mathfrak{p})$*. And the topology of* $Sper\ A$ *is given by the basis of open subsets*

$$U(a_1, ..., a_n) = \{(\mathfrak{p}, \alpha) \in Sper\ A : a_1(\alpha) > 0, ..., a_n(\alpha) > 0\}$$

where $a_1, ..., a_n \in A$*. This topology is known as the* ***Harrison topology****.*

Remark 3.4.9. *The subset* $U(a)$ *defines a sub-basis of open subsets.*

There are different ways to define the real spectrum, since we can describe the ordering of a ring by prime cone and homomorphism of ring into an ordered field.

Example 3.4.10. *The real spectrum of a field is simply the space of its orderings.*

Example 3.4.11. *The real spectrum of a real closed field is a point.*

Proposition 3.4.12. *Let* $\phi : A \to B$ *be a ring homomorphism. If* β *is a prime cone of* B*, then* $\phi^{-1}(\beta)$ *is a prime cone of* A*, and the mapping*

$$\begin{aligned} Sper(\phi) : Sper\ B &\longrightarrow Sper\ A \\ \beta &\longmapsto \phi^{-1}(\beta) \end{aligned}$$

is a continuous mapping. In the language of the category theory, $Sper$ *is a contravariant functor from the category of commutative rings with unit to the category of topological spaces.*

Proof. If $ab \in \phi^{-1}(\beta)$, then $\phi(a)\phi(b) \in \beta$, from the definition of the prime cone, or $\phi(a) \in \beta$ or $-\phi(b) \in \beta$. Hence, or $a \in \phi^{-1}(\beta)$ or $-b \in \phi^{-1}(\beta)$, this shows that $\phi^{-1}(\beta)$ is a prime cone. The continuity follows from the equality

$$(Sper(\phi))^{-1}(U(a_1, ..., a_n)) = U(\phi(a_1), ..., \phi(a_n)).$$

□

Proposition 3.4.13. *The support map given by*

$$\begin{aligned} supp : Sper\ A &\longrightarrow Sper\ B \\ \alpha &\longmapsto supp(\alpha) \end{aligned}$$

is a continuous map, whose image is the set of real prime ideals of A. In the language of the category theory, the support map is a natural transformation from the functor $Sper$ to the functor $Spec$.

Proof. Since $\mathfrak{p} \in Sper\ A$ is real if and only if its quotient field $\kappa(\mathfrak{p})$ is real, it is clear that the image of supp is the set of real prime ideals. Since $D(a) = \{\mathfrak{p} \in Spec\ A : a \notin \mathfrak{p}\}$ is a basic open subset of $Spec\ A$ and the pre-image

$$supp^{-1}(D(a)) = \{\alpha : a(\alpha) > 0 \text{ or } a(\alpha) < 0\} = U(a) \cup U(-a)$$

is the open subset of $Sper\ A$, the support map is continuous. □

For a real closed valuation ring, the real spectrum and the Zariski spectrum are homeomorphic.

Proposition 3.4.14. *If B is a real closed valuation ring, we have the support map $supp : Sper\ B \to Spec\ B$ is a homeomorphism.*

Proof. Since $supp$ is always continuous, it suffices to show that $supp$ is an open bijection. From Proposition 3.3.7, the map $supp$ is a bijection. Note that the ordering of the quotient field $\kappa(\mathfrak{p})$ of B is induced by the real closed field $Frac(B)$, so, if the image of $a \in B$ in any quotient field $\kappa(\mathfrak{p})$ is positive (resp. negative), then $a(\alpha) \geq 0$ (resp. $a(\alpha) \leq 0$) for all $\alpha \in Sper\ B$. Hence, $supp(U(a)) = D(a)$ or $supp(U(a)) = \varnothing$, since both are open subset of $Spec\ B$, the map $supp$ is open. □

Remark 3.4.15. *Every real spectrum is homeomorphic to a Zariski spectrum, since the real spectrum is a spectral space.*

Proposition 3.4.16. *Let R be a closed real field, and V an algebraic set in R^n. Then the function $\phi : V \to Sper(R[V])$, defined by $z \mapsto P_z = \{f \in R[V] : f(x) > 0\}$, is injective and induces a homeomorphism from V (with Euclidean topology) to $\phi(V)$.*

Proof. The subset P_x forms a prime cone of $R[V]$: $f(x)g(x) > 0$ implies that $f(x) > 0$ or $-g(x) > 0$. It is clear that if $x \neq y$, then $P_x \neq P_y$. Let $U(f_1, ..., f_m)$ be a basic open subset. Then, we have

$$E(f_1, ..., f_m) := \phi^{-1}(U(f_1, ..., f_m)) = \{x \in V : f_1(x) > 0, ..., f_m(x) > 0\},$$

which forms a basis for the Euclidean topology on V. Moreover, by definition,

$$\phi(E(f_1, ..., f_m)) = U(f_1, ..., f_m) \cap \phi(V).$$

Therefore, ϕ is injective and induces a homeomorphism. □

This is a very interesting result; we not only have an embedding, but also an embedding with the Euclidean topology.

Any point of an algebraic set over an algebraically closed field corresponds bijectively to a closed point in its associated Zariski spectrum. However, this does not hold in the real spectrum. For example, the set of closed points of $Sper(\mathbb{R}[X])$ is $\mathbb{R} \cup \{-\infty, +\infty\}$ (for more details, see Example 7.1.4 of [3]).

Let R be a real closed field. An R-valued point of $X := Spec\ A$ is a morphism of schemes $Spec\ R \to Spec\ A$, which corresponds to a homomorphism $A \to R$ of rings. From the third equivalent condition for ordering in a ring, the real spectrum $Sper\ A$ can be interpreted as the collection of real points $(\bigcup_R X(R))/\sim$ of X quotient by an equivalent relation, where R run through all real closed fields.

Hence, if $Sper\ A = \varnothing$, then the Zariski spectrum has no real points. The converse holds when A is an integral domain that is a finite-type R-algebra, where R is a real closed field.

Proposition 3.4.17. *Let R be a real closed field, and let A be an integral domain that is a finite-type R-algebra. Then, $Sper\ A = \varnothing$ if and only if the set of R-valued points $(Spec\ A)(R) = \varnothing$.*

Proof. In this case A corresponds to an irreducible R-algebraic set V (i.e., $A = R[V]$), and an R-valued point $Spec\ R \to Spec\ A$ corresponds to a ring homomorphism $\phi : A \to R$, which represents a point $(\phi(X_1), ..., \phi(X_n))$ of V. Therefore, it suffices to prove that $V = \varnothing$ if and only if $Sper\ R[V] = \varnothing$.

Assume $Sper\ R[V] = \varnothing$. The previous proposition ensures that there exists an injective map $V \hookrightarrow Sper\ R[V]$. Therefore, we conclude that $V = \varnothing$.

Assume $V = \varnothing$, by the Real Nullstellensatz, for any prime ideal $\mathfrak{p}$, we have

$$\sqrt[R]{\mathfrak{p}} = I(V(\mathfrak{p})) = I(\varnothing) = R[V] \neq \mathfrak{p},$$

which implies that $\mathfrak{p}$ is not real. Hence, by the second equivalence of points in the real spectrum, we obtain $Sper\ R[V] = \varnothing$.

□

The real spectrum can be generalized to a scheme X. The natural way to do this is by glueing together the local real spectra, as detailed below:

Let $X = \bigcup_{i \in I} Spec\ A_i$ be a scheme, where $Spec\ A_i$ forms an open affine covering of X, the real spectrum $Sper\ \mathcal{O}_X(Spec\ A_i) = Sper\ A_i$ can be glued together. The resulting topological space is called the **real spectrum** of X, denoted by X_r.

Now we describe the real spectrum of a scheme and show that it is independent of the choice of covering. Let $\mathfrak{p}$ be a point in X, by the definition of scheme, the point $\mathfrak{p}$ belongs to an open affine subscheme $\mathfrak{p} \in Spec\ A \subseteq X$. Hence, $\mathfrak{p}$ corresponds to a prime ideal of A, and the quotient field $\kappa(\mathfrak{p})$ coincides with the residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = (A/\mathfrak{p})_{\mathfrak{p}A/\mathfrak{p}} = Frac(A/\mathfrak{p})$ (since localization is an exact functor). This means that a point $(\mathfrak{p}, \alpha) \in Sper\ \mathcal{O}_X(Spec\ A) = Sper A$ corresponds bijectively to an ordering of the residue field $\kappa(\mathfrak{p})$ of $\mathcal{O}_{X,\mathfrak{p}}$. Therefore, each point of X_r can be identified as a pair (x, α) where $x \in X$ and α is a positive cone/ordering of the residue field $\kappa(x)$ of $\mathcal{O}_{X,x}$, which is independent of the choice of covering.

Definition 3.4.18. *Let X be a scheme, the real spectrum of X, denoted by X_r is a topological space consisting of pairs (x, α), where $x \in X$ and α is a positive cone of the residue field $\kappa(x)$. The topology of X_r is given by the basis of open sets*

$$U(a_1, ..., a_n) = \{(\mathfrak{p}, \alpha) \in Sper\ A : a_1(\alpha) > 0, ..., a_n(\alpha) > 0\}$$

where A is the ring corresponding to an open affine subscheme Spec A of X and $a_1, ..., a_n \in A$.

Since the support map is continuous in affine case, we have

Proposition 3.4.19. *The support map given by*

$$\begin{aligned} supp : X_r &\longrightarrow X \\ (\mathfrak{p}, \alpha) &\longmapsto \mathfrak{p} \end{aligned}$$

is continuous.

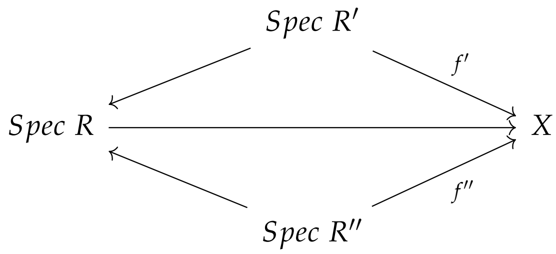
The functor $(\ \ )_r$ is also a covariant functor from the category of schemes to the category of topological spaces.

Proposition 3.4.20. *If $f : Y \to X$ is a morphism of schemes, then f induces a continuous map between the real spectral*

$$\begin{aligned} f_r : Y_r &\longrightarrow X_r \\ (y, \alpha) &\longmapsto (f(y), \overline{f_y^{\#}}^{-1}(\alpha)) \end{aligned}$$

where $\overline{f_y^\#} : \kappa(f(y)) \to \kappa(y)$ *is a homomorphism induced by the map* $f_y^\# : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ *of fibers/stalks.*

Just as with the real spectrum of rings, we also have a categorical description of X_r : The elements of X_r can be identified with equivalence classes of morphisms of schemes $f : Spec\ R \to X$, where R is a real closed field. Two morphisms $f' : Spec\ R' \to X$ and $f'' : Spec\ R'' \to X$ lie in the same equivalence class if and only if there exists a commutative diagram



where R is also a real closed field.

Chapter 4

Specialization

In this chapter, we introduce the concept of specialization in the real spectrum of schemes, which will be an important tool for our subsequent discussion. Readers who are willing to accept the final results presented in later sections may choose to skip this chapter.

In the first section, we present some basic facts about specialization in arbitrary topological spaces. In the second section, we discuss specialization in the real spectrum. Finally, at the end of the chapter, we provide a categorical description of specialization in real spectrum. The references for this chapter are [3] and [27].

4.1 Specialization in a topological space

In topology, the specialization relation provides a way to examine the closeness or "relation" between points in a space, particularly in spaces where the separation properties (like those of T1 spaces) are weaker.

Definition 4.1.1. *1. Let X be a topological space and x, y be two points of X. We say that y is a* ***specialization*** *of x or x is a* ***generalization*** *of y, denoted by $x \rightsquigarrow y$, if $y \in \overline{\{x\}}$ (closure of $\{x\}$).*

2. A subset $T \subseteq X$ is ***stable under specialization*** *if for any $x' \in T$ and every specialization $x' \rightsquigarrow x$ we have $x \in T$.*

3. A subset $T \subseteq X$ is ***stable under generalization*** *if for any $x \in T$ and every specialization $x' \rightsquigarrow x$ we have $x' \in T$.*

Definition 4.1.2. *Let $f : X \to Y$ be a function of topological spaces. We say f* ***preserves specialization (or generalization)*** *if for any $x, x' \in X$,*

$$x' \rightsquigarrow x \quad \Rightarrow \quad f(x') \rightsquigarrow f(x).$$

Proposition 4.1.3. *The specialization relation is an ordering relation.*

Proof. The specialization relation is reflexive, since $x \in \overline{\{x\}}$. If $x \in \overline{\{x'\}}$ and $x' \in \overline{\{x\}}$, then we have $\overline{\{x\}} \subseteq \overline{\{x'\}}$ and $\overline{\{x'\}} \subseteq \overline{\{x\}}$, therefore, the specialization relation is anti-symmetric. If $x \in \overline{\{x'\}}$ and $x' \in \overline{\{x''\}}$, then $x \in \overline{\{x'\}} \subseteq \overline{\{x''\}}$, which shows that the relation is transitive. □

Proposition 4.1.4. *Let X be a topological space.*

1. *Every closed subset of X is stable under specialization.*

2. *A subset $T \subseteq X$ is stable under specialization if and only if the complement $X \setminus T$ of T is stable under generalization.*

3. *Every open subset of X is stable under generalization.*

Proof. Statement (1) is immediate from the definition. Statement (2) can be easily proven by contrapositive, and statement (3) follows from (1) and (2) by considering the complement. □

Definition 4.1.5. *Let $f : X \to Y$ be a continuous function of topological spaces.*

1. *The function f is **generalizing** if, for any $y' \rightsquigarrow y$ in Y and $x \in X$ with $f(x) = y$, there exists a generalization $x' \rightsquigarrow x$ of x in X such that $f(x') = y'$.*

2. *The function f is **specializing** if, for any $y' \rightsquigarrow y$ in Y and $x' \in X$ with $f(x') = y'$, there exists a specialization $x' \rightsquigarrow x$ of x' in X such that $f(x) = y$.*

Proposition 4.1.6. *Let $f : X \to Y$ be a continuous map of topological spaces.*

1. *If f is generalizing, and if $T \subseteq X$ is stable under generalization, then $f(T) \subseteq Y$ is stable under generalization.*

2. *If f is specializing, and if $T \subseteq X$ is stable under specialization, then $f(T) \subseteq Y$ is stable under specialization.*

Proof. Let $y' \rightsquigarrow y$ be a specialization in Y where $y \in f(T)$. So there is an $x \in T$ such that $f(x) = y$. By the definition of generalizing, there exists a generalization $x' \rightsquigarrow x$ of x in X such that $f(x') = y'$. Since T is stable under specialization, $x \in T$, and then $y \in f(T)$. Therefore $f(T)$ is stable under specialization.

The proof of the other statements is identical. □

4.2 Specialization in the real spectrum

Specialization plays an important role in the study of the real spectrum, but we will not discuss it systematically; instead, we will focus on selected facts that we will use later.

Proposition 4.2.1. *Let ξ', ξ be two points of $Sper\ A$, where A is a commutative ring with unit. The following conditions are equivalent:*

1. *$\xi' \rightsquigarrow \xi$.*
2. *For every $a \in A$, if $a >_{\xi} 0$, then $a >_{\xi'} 0$.*
3. *For every $a \in A$, if $a \geq_{\xi'} 0$, then $a \geq_{\xi} 0$.*
4. *If we identify points of the real spectrum with prime cones, then $\xi' \subseteq \xi$.*

Proof. The last three statements are clearly equivalent. Statements (1) and (2) are both equivalent to

$$\xi \in U(a_1, ..., a_n) \Rightarrow \xi' \in U(a_1, ..., a_n),$$

so they are also equivalent. □

Remark 4.2.2. *If $\xi' \rightsquigarrow \xi$, then $supp(\xi') = \xi' \cap -\xi' \subseteq \xi \cap -\xi = supp(\xi)$. Thus, there is a canonical map $A_{supp(\xi)} \to A_{supp(\xi')}$. In general, this homomorphism is not a local homomorphism.*

Remark 4.2.3. *Every Zariski spectrum of a local domain has a generic point and a closed point.*

Lemma 4.2.4. *Let A be a local domain, and let ξ', ξ be two points of $Sper\ A$. Let B be the convex hull of A in $\kappa(\xi')$. If $\xi' \rightsquigarrow \xi$ and the support of ξ' (resp. ξ) is the generic point (resp. closed point) of $Spec\ A$, then $m_A \subseteq m_B$. In particular, there is a homomorphism $(A/m_A, \xi) \to B/m_B$ that preserves the ordering.*

Proof. Suppose there exists $x \in m_A \setminus m_B$. Since B is a valuation ring (and in particular, a local domain), we know that $x^{-1} \in B$. Since B is the convex hull of A in $\kappa(\xi')$, there exists $y \in A$ such that $\frac{1}{|x|} \leq_{\xi'} y$. In other words, we have $1 \leq_{\xi'} |x|y$.

By the previous proposition, it follows that $1 \leq_{\xi} |x|y$. However, $|x|y \in m_A$, implying that the image of $|x|y$ in $\kappa(supp(\xi)) = A/m_A$ is zero. This leads to a contradiction.

Therefore, we conclude that there exists a canonical local homomorphism $A \to B$, which induces a homomorphism between the fields $(A/m_A, \xi) \to B/m_B$. This homomorphism preserves the ordering, since ξ' is a generalization of ξ. □

Proposition 4.2.5. *Let α be an element of $Sper\ A$. The specializations of α are totally ordered with respect to the specialization. Specifically, if $\alpha \rightsquigarrow \beta$ and $\alpha \rightsquigarrow \gamma$, then either $\beta \rightsquigarrow \gamma$ or $\gamma \rightsquigarrow \beta$.*

Proof. Suppose the conclusion is false. Then there exist elements $b \in \beta \setminus \gamma$ and $c \in \gamma \setminus \beta$. By the definition of the prime cone, we know that either $b - c \in \alpha$ or $c - b \in \alpha$. In the first case, we have $b = c + (b - c) \in \gamma$, while in the second case $c = b + (c - b) \in \beta$. In both cases, we arrive at a contradiction, so the conclusion must be true. □

The following result will be used in the next chapter. We do not introduce the notion of constructible sets here, as it appears only once in this thesis. This is just a brief comment; for more details, see Chapter 7 of [3].

Theorem 4.2.6. *Let C be a constructible sets of $Sper\ A$. Then C is a closed (resp. open) subset if and only if it is stable under specialization (resp. generalization).*

Proof. See 7.1.22 of [3]. □

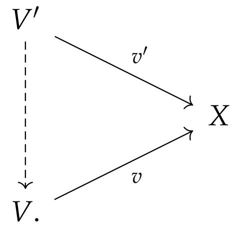
Remark 4.2.7. *Every basic open set in $Sper\ A$ is constructible and thus stable under generalization. If $f : Sper\ A \to Sper\ B$ is generalizing and maps constructible subsets to constructible ones, from Proposition 4.1.6, $f(U)$ is constructible and stable under generalization for every basic open U of $Sper\ A$. Hence, $f(U)$ is an open subset of $Sper\ B$, which means that f is an open map.*

If a real spectrum has a generic point, then it is totally ordered by the specialization relation. Since every local domain admits both a closed point and a generic point, and since the real spectrum and the Zariski spectrum of a real closed valuation ring are homeomorphic, both spectrums have a closed point and a generic point, and are totally ordered by the specialization relation. This allows us to describe the specialization in the real spectrum in terms of the real closed valuation rings.

Definition 4.2.8. *Let $v : V \to X$ be a morphism of schemes, where V is spectrum of a real closed valuation ring. The **specialization in** X_r **determined by** v is given by $v_r(\zeta') \rightsquigarrow v_r(\zeta)$, where ζ' (resp. ζ) is the generic point (resp. closed point) of V_r.*

Remark 4.2.9. *Since v_r is continuous, we have $v_r(\overline{\{\zeta'\}}) \subseteq \overline{v_r(\{\zeta'\})}$, and $\overline{v_r(\{\zeta'\})}$ is an irreducible component.*

Any specialization $\xi' \rightsquigarrow \xi$ in X_r is determined in this way. In fact there is a unique minimal choice of v, i.e., there is a morphism $v : V \to X$, with V is the spectrum of a real closed valuation ring, such that v determines the specialization $\xi' \rightsquigarrow \xi$. Furthermore, if $v' : V' \to X$ is another morphism that determines $\xi' \rightsquigarrow \xi$, then v' factors uniquely as



Existence: Consider an affine open neighborhood $U = Spec(A)$ of $supp(\xi)$. Since every open set is stable under generalization, $supp(\xi') \in U$. We obtain the ring homomorphism

$$A \to A_{supp(\xi)} = \mathcal{O}_{X,supp(\xi)}.$$

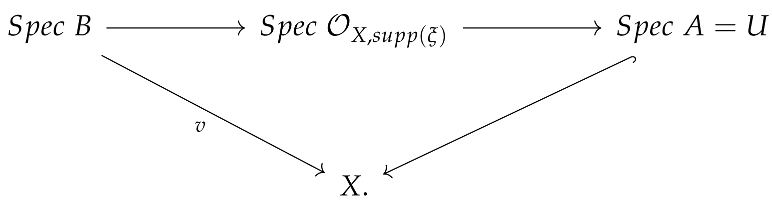
By the Proposition 4.2.1, the specialization $\xi' \rightsquigarrow \xi$ implies that $supp(\xi') \subseteq supp(\xi)$. Therefore, we have a ring homomorphism $A_{supp(\xi)} \to A_{supp(\xi')}$. This gives a sequence of homomorphisms

$$\mathcal{O}_{X,supp(\xi)} = A_{supp(\xi)} \to \kappa(supp(\xi')) \to \kappa(\xi').$$

Let B be the convex hull of the image $\overline{\mathcal{O}_{X,supp(\xi)}}$ of $\mathcal{O}_{X,supp(\xi)}$ in $\kappa(\xi')$. We then obtain the following sequence of ring homomorphisms:

$$A \to \mathcal{O}_{X,supp(\xi)} \to \overline{\mathcal{O}_{X,supp(\xi)}} \to B \tag{4.1}$$

which induces a morphism $v : Spec\ B \to X$



We need to show that the specialization $\xi' \rightsquigarrow \xi$ is determined by v. From the diagram above, it is clear that $v(0) = supp(\xi')$. Now, we prove that $v(m_B) = supp(\xi)$: From the diagram above, it suffices to prove that the image $\overline{supp(\xi)}$ of $supp(\xi)$ in $\overline{\mathcal{O}_{X,supp(\xi)}}$ is contained in m_B. Suppose, for the sake of contradiction, that there exists some $\overline{x} \in \overline{supp(\xi)} \setminus m_B$. Since B is a local domain, we have $\overline{x}^{-1} \in B$. By the definition of convex hull, there exists an element $\frac{\overline{p}}{\overline{q}} \in \overline{\mathcal{O}_{X,supp(\xi)}}$ such that $\overline{q} >_{\xi'} 0$ and $|\overline{x}|^{-1} \leq_{\xi'} \frac{\overline{p}}{\overline{q}}$. This implies that $0 <_{\xi'} \overline{q} \leq_{\xi'} |\overline{x}|\overline{p}$. Since the homomorphism $\mathcal{O}_{X,supp(\xi)} \to \overline{\mathcal{O}_{X,supp(\xi)}}$ is surjective, there exist elements $|x| \in supp(\xi)$, and $p, q \in A$ such that $q \notin supp(\xi)$ correspond to $|\overline{x}|$, $\overline{p}$, and $\overline{q}$. By Proposition 4.2.1, we know that $0 \leq_{\xi} q \leq_{\xi} |x|p$. However, since $|x|p \in supp(\xi)$ and $q \notin supp(\xi)$, this leads to the contradiction $0 <_{\xi} q \leq 0$. Thus, we conclude that $\overline{x} \in m_B$, and therefore, $\overline{supp(\xi)} \subseteq m_B$. This implies that $v(m_B) = supp(\xi)$, as desired.

Since $Sper\ B \cong Spec\ B$ and the ordering on B is induced by $\kappa(\xi')$, we have $v_r(supp^{-1}(0)) = \xi'$ and $v_r(supp^{-1}(m_B)) = \xi$, as desired.

Minimality: Let $v' : V' \to X$ be a morphism of schemes that determines the specialization $\xi' \rightsquigarrow \xi$, where $V' = Spec\ B'$ for some real-closed valuation ring B'. Since any open subset is stable under generalization and B' has a generic point, the morphism v' factors as

$$V' \to Spec\ A \hookrightarrow X$$

where $Spec\ A$ is an open affine subscheme of X containing ξ'. Let $\varphi : A \to B'$ be the ring homomorphism corresponding to the morphism $V' \to Spec\ A$. Since $v'(m_{B'}) = supp(\xi)$, by the universal property of localization, the map $A \to B'$ factors through

$$A \to \mathcal{O}_{X,supp(\xi)} \to B'.$$

Since v' determines the specialization $\xi' \rightsquigarrow \xi$, we have $\varphi^{-1}(0) = supp(\xi')$. Applying the isomorphism theorem, we obtain a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & \mathcal{O}_{X,supp(\xi)} & \longrightarrow & \text{Im } \varphi & \longrightarrow & B'. \\ & \searrow & \downarrow & \nearrow\swarrow & & & \\ & & \mathcal{O}_{X,supp(\xi)}/supp(\xi')\mathcal{O}_{X,supp(\xi)} & & & & \end{array}$$

From the construction of v, we have Im $(\varphi) \cong \mathcal{O}_{X,supp(\xi)}/supp(\xi')\mathcal{O}_{X,supp(\xi)} \cong \overline{\mathcal{O}_{X,supp(\xi)}}$. Let f be the isomorphism between $\overline{\mathcal{O}_{X,supp(\xi)}}$ and Im (φ). Now, let $conv(\text{Im } \varphi)$ denote the convex hull of Im (φ) in the quotient field $Frac(B') = \kappa(\xi')$ (this equality holds, because $v_r'(supp^{-1}(m_B)) = \xi'$). Since the convex hull is the smallest convex ring containing Im(φ), there exists a inclusion

$$\begin{array}{ccccccc} A & \longrightarrow & \mathcal{O}_{X,supp(\xi)} & \longrightarrow & \text{Im } \varphi & \longrightarrow & B'. \\ & & & & \big\downarrow & \nearrow & \\ & & & & conv(\text{Im } (\varphi)) & & \end{array}$$

Since $\mathrm{Im}(\phi) \cong \overline{\mathcal{O}_{X,supp(\xi)}}$, we expect that $conv(\mathrm{Im}\ (\varphi))$ is isomorphic to B. The isomorphism

$$f : \overline{\mathcal{O}_{X,supp(\xi)}} \to \mathrm{Im}\ (\varphi)$$

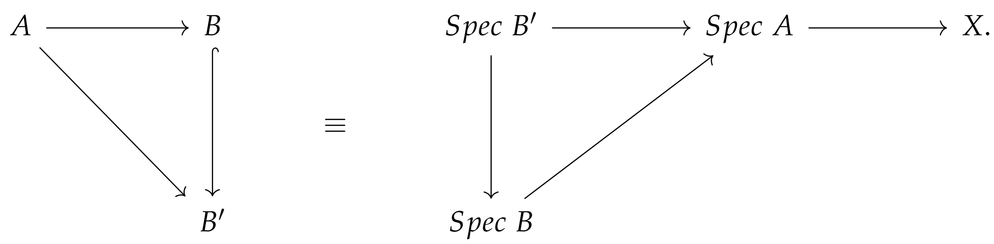
can be extended to an isomorphism

$$f_0 : \kappa(supp(\xi')) \to Frac(\mathrm{Im}\ (\varphi))$$

of fields of fractions. Since the isomorphism f_0 preserves ordering by construction, it induces an isomorphism g between the real closures of these fields, namely $\kappa(\xi')$.

Finally, the restriction $\overline{f}$ of g to B is an order-preserving isomorphism between B and $conv(\mathrm{Im}\ (\varphi))$: By the definition of convex hull, for every $x \in conv(\mathrm{Im}\ (\varphi))$, there exist $a, b \in \mathrm{Im}(\varphi)$ such that $a \leq x \leq b$. This implies that $f^{-1}(a) \leq \overline{f}^{-1}(x) \leq f^{-1}(b)$, so $\overline{f}^{-1}(x) \in \mathrm{Im}\ (\varphi)$. Hence, $conv(\mathrm{Im}(\varphi)) \subseteq \mathrm{Im}(\overline{f})$. Conversely, by a similar reasoning, we can show that $conv(\mathrm{Im}(\varphi)) \supseteq \mathrm{Im}(\overline{f})$.

This establishes the existence of the following commutative diagram



Uniqueness: This is straightforward to prove.

Lemma 4.2.10. *Let $f : Y \to X$ be a morphism of schemes, and suppose that for every $y \in Y$ the field extension $\kappa(y) \supseteq \kappa(f(y))$ is algebraic (so, f could be étale).*

1. *If $\xi \in X_r$ is represented by a morphism of schemes, $\alpha : Spec\ R \to X$, where R is a real closed field, then the natural map*

$$hom_{Scheme/X}(Spec\ R, Y) \longrightarrow Y_r \tag{4.2}$$

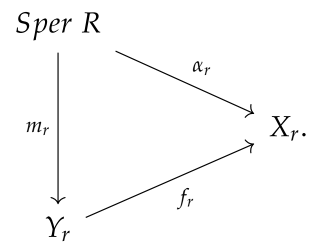
$$h \longmapsto (h(*), \alpha_h) \tag{4.3}$$

   (where α_h denotes the ordering induced by h) is a bijection from the set on the left to $f_r^{-1}(\xi)$, the real spectrum fiber of $\xi \in Y_r$. Here, $*$ denotes the unique prime ideal 0 of R.

2. *Let $v : V \to X$ be a morphism of schemes, where V is the spectrum of a real closed valuation ring. Suppose that a specialization $\eta' \rightsquigarrow \eta$ in Y_r is given, such that the specialization $f_r(\eta') \rightsquigarrow f_r(\eta)$*

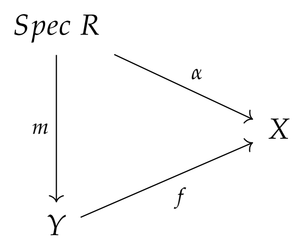
in X_r is the specialization determined by v. Then there is a unique X-morphism $V \to Y$ that determines $\eta' \rightsquigarrow \eta$.

Proof. (1) : If $f_r^{-1}(\xi) = \varnothing$, then there does not exist a morphism $m : Spec\ R \to Y$ that makes the following diagram commute.

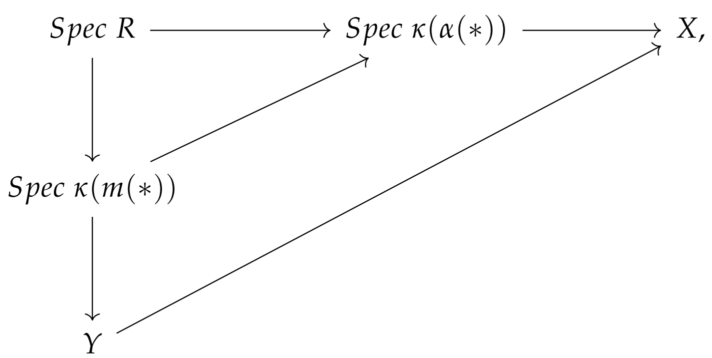


Therefore, $hom_{Scheme/X}(Spec\ R, Y) = \varnothing$.

We suppose that $f_r^{-1}(\xi) \neq \varnothing$, in particular $f^{-1}(supp(\xi)) \neq \varnothing$. Since the diagram

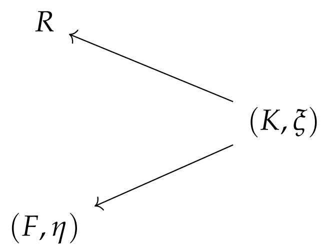


factors through

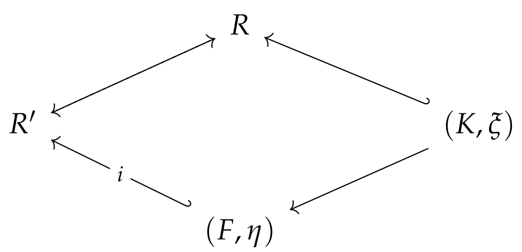


we can reduce to the case where X and Y are Zariski spectrums of fields. Assume that R is the real closure of the field associated with $\xi \in X_r$. Let $X = Spec\ K$, and $Y = Spec\ F$. Note that for each $\eta \in f_r^{-1}(\xi)$, the map $Spec^{-1}(f)$ is order-preserving. We first prove that for a given $\eta \in f_r^{-1}(\xi)$, there exists a morphism $Spec\ R \to Y$ such that $\alpha(Spec\ R \to Y) = \eta$. Since R is a real closed field, this is equivalent to finding a homomorphism from (F, η) to R.

The morphism f and α induce the homomorphisms of rings.



Let R' be a real closure of (F, η), and denote the inclusion by $i : (F, \eta) \hookrightarrow R'$. Since $Spec^{-1}(f)$ is an ordering-preserve map, and $F : K$ is an algebraic extension, we have that the real closure R' of (F, η) is also a real closure of (K, ξ).



Hence, by Proposition 3.1.20, there exists a unique isomorphism ϕ from R to R' that extends $i \circ Spec^{-1}(f)$. Therefore, there is a homomorphism from (F, η) to R; in other words, the map 4.2 is surjective onto $f_r^{-1}(\xi)$.

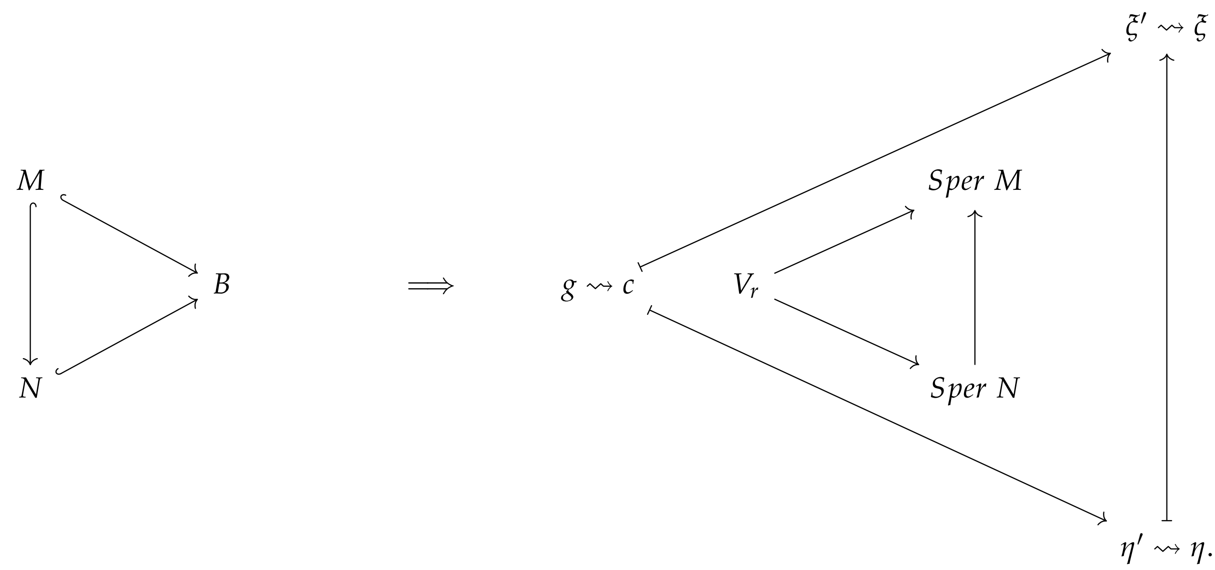
It remains to show that the map 4.2 is injective. Suppose that $m_1, m_2 : F \to R$ are induced by X-morphisms and that $m_1^{-1}(R^{(2)}) = m_2^{-1}(R^{(2)})$, meaning they induce the same ordering. Then, m_1 and m_2 factor through $F \to R' \to R$, where R' is the real closure of $(F, m_1^{-1}(R^{(2)}))$. The uniqueness of isomorphisms between real closures (Theorem 3.1.19) ensures that $m_1 = m_2$.

(2) : We begin by proving uniqueness. Suppose $v_1, v_2 : V = Spec\ B \to Y$ take work. Define two new morphisms $\tilde{v_1}, \tilde{v_2} : Spec\ (Frac(B)) \to V \to Y$, where the map $Spec\ (Frac(B)) \to V$ is induced by the inclusion. By the hypothesis, we have $\tilde{v_i}(0) = v_i(0)$ for $i = 1, 2$. However, by item (1), we know that $\tilde{v_1} = \tilde{v_2}$, which implies that $v_1(0) = v_2(0)$. Since 0 is the generic point, $v_1 = v_2$ as function. The equality of the sheaf maps follows from $\tilde{v_1} = \tilde{v_2}$.

It remains to show the existence. Since $V = Spec\ B$ is the spectrum of a real closed valuation ring. As in the previous argument, we reduce to the case where $X = Spec\ M$, $Y = Spec\ N$ with M and N being local domains, where f is induced by a local inclusion $M \subseteq N$, and the support of η' (resp. η) corresponds to the generic point (resp. the closed point) of Y.

Let $\xi' := f_r(\eta')$ and $\xi := f_r(\eta)$. Let K be the field of fractions of M, L the field of fractions of N, and let $L \to R$ be the real closure of L with respect to η'. Thus $K \subseteq L \hookrightarrow R$ is the real closure of

K with respect to ξ'. Without loss of generality, suppose that v is minimal for the specialization $\xi' \rightsquigarrow \xi$, i.e., B is the convex hull of M in R (with v induced by $M \subseteq B$). Therefore, what remains to be shown is that $N \subseteq B$, which induces the diagram on the right below:



Let $C \supseteq B$ be the convex hull of N in R. Consider the sequence of homomorphisms

$$M \xrightarrow{(i)} \kappa(\xi) \xrightarrow{(ii)} \kappa(\eta) \xrightarrow{(iii)} \kappa_C \tag{4.4}$$

where $\kappa(\xi)$ (resp. $\kappa(\eta)$) is a real closure of residue field κ_M (resp. κ_N) of M (resp. N), and κ_C is the residue field of C. We now explain the morphisms in the positions (i), (ii) and (iii) :

(i) : The morphism is induced by the canonical map $M \to \kappa_M \hookrightarrow \kappa(\xi)$;

(ii) : Since $\kappa_M \subseteq \kappa_N$ is an algebraic field extension, we have a κ_M-isomorphism $\kappa(\xi) \to \kappa(\eta)$ (by Proposition 3.1.20);

(iii) : By Lemma 4.2.4, there exists a homomorphism that preserves the ordering $\kappa_N \to \kappa_C$. Since C is a real closed valuation ring, κ_C is a real closed field. So, by Proposition 3.1.20, there exists an extension $\kappa(\eta) \to \kappa_C$.

Since κ_M is archimedean over the image of M, and $\kappa(\xi)$ is also archimedean over the image of κ_M (by the archimedean property of real closures 3.3.10), it follows that $\kappa(\xi)$ is archimedean over the image of M. Furthermore, since (ii) is an isomorphism, $\kappa(\eta)$ is archimedean over the image of M. The field κ_C is also archimedean over the image of M, since κ_C is the convex hull of N.

Finally, since the image of M in κ_C is contained in B/m_C, and B is the convex hull of M in R, it follows that $B = C$. □

The first part of the lemma provides a categorical perspective on the ordering induced by a morphism of schemes, while the second part guarantees that the specialization relation has the lifting property.

With this lemma, we conclude this chapter, which provides the technical groundwork for the next one.

Chapter 5

Real Étale Topos

In this chapter, we introduce another Grothendieck topology, the real étale site X_{ret}, and prove that the real étale topos, the category of sheaves on this site, is equivalent to the category of sheaves on the associated real spectrum. The primary reference for this chapter is [27].

In the first section, we use the tools developed in the previous chapter to show that the functor $(\ )_r$ preserves pullbacks in Et/X, and that if $f : X \to Y$ is an étale morphism, then f_r is a local homeomorphism.

In the second section, we prove that the topos $Sh(X_r)$ is equivalent to the topos $Sh(X_{ret})$.

5.1 Real étale site and real spectrum

We now define the real étale site of a scheme X and its Grothendieck topos

Definition 5.1.1. *A family $\{f^i : U^i \to U\}_{i\in I}$ of morphisms of schemes is said to be **real surjective** if and only if*

$$U_r = \bigcup_{i\in I} f_r^i(U_r^i).$$

It is easy to see that the collection of real surjective families forms a Grothendieck topology.

Definition 5.1.2. *Let X be a scheme. The topology on Et/X defined by the real surjective families is called the **real étale topology** of X, abbreviated "ret". The site $(Et/X,\ ret)$ is called the **real étale site** of X and is denoted by X_{ret}. The category $Sh(X_{ret})$ of sheaves on X_{ret} is called the **real étale topos** of X.*

Before continuing, I would like to recall some facts about Henselian rings.

Remark 5.1.3. *1. Let A be a local ring, $X = Spec\ A$, and let x be the unique closed point of X. The ring A is Henselian, if and only if, for any étale morphism $f : Y \to X$ and for every point $y \in Y$ such that $f(y) = x$ and the residue fields satisfy $\kappa(y) = \kappa(x)$, there exists a section $s : X \to Y$ to f, that is, $f \circ s = id_X$. (See Lemma 10.153.3., Chapter I of [31])*

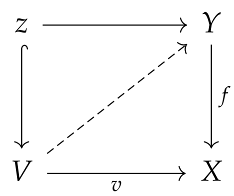
2. *Every real closed valuation ring is Henselian ([7]).*

The étale morphism is the analogue of a local homeomorphism in topology, and the map $(\ )_r$ reflects this analogy. Specifically, if f is an étale morphism, then the f_r is a local homeomorphism.

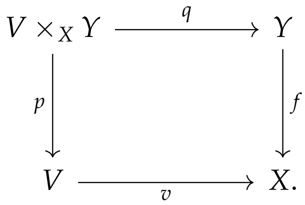
Lemma 5.1.4. *For any étale morpism $f : Y \to X$, the map $f_r : Y_r \to X_r$ is open.*

Proof. Since a locally open map is open, we may assume without loss of generality that X and Y are affine. In this case, the morphism f is finitely presented. A consequence of the Tarski-Seidenberg theorem is that if f is finitely presented, then the map f_r sends constructible sets to constructible sets (see [8] Prop. 2.3). Therefore, it suffices to show that f_r is generalizing (see Remark 4.2.7), that is, given $\eta \in Y_r$ and $\xi' \in X_r$ such that $\xi' \rightsquigarrow f_r(\eta)$, we need to find $\eta' \in Y_r$ such that $\eta' \rightsquigarrow \eta$ and $f_r(\eta') = \xi'$.

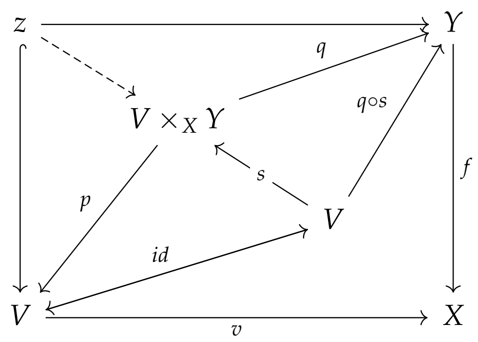
Let $\xi' \rightsquigarrow f_r(\eta)$ be represented by a morphism $v : V \to X$, where $V = Spec\ R$ is the spectrum of a real closed valuation ring R. Let $z \hookrightarrow V$ denote the inclusion of the closed point of V. Using Lemma 4.2.10, we have a commutative diagram



In the commutative diagram, $z \to Y$ represents the point η. It is sufficient to show that the dotted lift exists, leaving the diagram commutative. To do this, consider the pullback of the pair (v, f)



Since f is étale, its base change p is also étale, Let $w = z \otimes_X supp(\eta) \in V \times_X Y$ be a point. Then we have $p(w) = z$. Since p is étale, the residue field $\kappa(w)$ is algebraic over $\kappa(z)$. The residue field $\kappa(z)$ of a real closed valuation ring R is real closed, and since $\kappa(w)$ inherits an ordering induced by η and $\kappa(z)$, we have $\kappa(w) = \kappa(x)$. By the Henselian property of real closed valuation rings, there exists a section $s : V \to V \times_X Y$ of p. Thus, we have a commutative diagram as follows:



□

Proposition 5.1.5. *Let the following square*

$$\begin{CD} Y \times_X Z @>{q}>> Z \\ @V{p}VV @VV{g}V \\ Y @>{f}>> X \end{CD}$$

be a pullback (fiber product) of schemes, and assume that f and g are étale. Then the natural map

$$\begin{aligned} \gamma : (Y \times_X Z)_r &\longrightarrow Y_r \times_{X_r} Z_r \\ (I, \alpha) &\longmapsto (p_r(I, \alpha), q_r(I, \alpha)) \end{aligned}$$

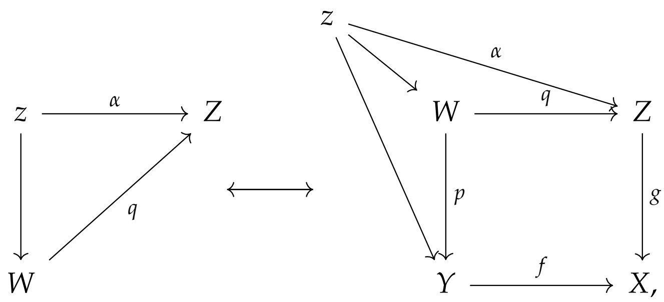
is a homeomorphism (the right hand side is the pullback in the sense of topological spaces).

Proof. Write $W := Y \times_X Z$. Note that q and p are also étale, since they are base changes of étale morphism. From Lemma 4.2.10, it follows that γ is bijective. Specifically, if $\zeta \in Z_r$ is represented by $\alpha : z \to Z$, with z the spectrum of a real closed field, then the diagram

$$\begin{CD} hom_{Scheme/Z}(z, W) @<>> hom_{Scheme/X}(z, Y) \\ @VVV @VVV \\ q_r^{-1}(\zeta) @>{\gamma}>> f_r^{-1}(g_r(\zeta)) \end{CD}$$

commutes. The top horizontal arrow is obtained by composing the Z-morphism from z to W

with the map p



and its bijectivity follows from the universal property of the pullback. The vertical arrows are the maps given by Lemma 4.2.10. Since

$$\begin{aligned} f_r^{-1}(g_r(\zeta)) &= \{\eta \in Y_r : f_r(\eta) = g_r(\zeta)\} \\ &= Y_r \times_{X_r} \{\zeta\}, \end{aligned}$$

the map γ is a bijection.

The map γ is continuous and open because the projections

$$pr_1(\gamma(I,\alpha)) = p_r(I,\alpha) \text{ and } pr_2(\gamma(I,\alpha)) = q_r(I,\alpha)$$

are continuous and open (by Lemma 5.1.4). Therefore, γ is a homeomorphism. □

Corollary 5.1.6. *For any scheme X, the real spectrum functor*

$$\begin{aligned} (\ \ )_r : Et/X &\longrightarrow Top/X_r \\ U &\longmapsto U_r \end{aligned}$$

preserves pullbacks, and therefore it preserves all finite inverse limits. In particular, $(\ \ )_r$ preserves monomorphisms.

Proposition 5.1.7. *For any étale morpism $f : Y \to X$, the map $f_r : Y_r \to X_r$ is a local homeomorphism.*

Proof. By Lemma 5.1.4, the map f_r is open. Since f is étale (and in particular, unramified), the diagonal map $Y \to Y \times_X Y$ is an open immersion (by Proposition 2.1.9), it factors through

$$Y \cong U \hookrightarrow Y \times_X Y,$$

where U is an open subscheme of $Y \times_X Y$. Corollary 5.1.6 ensures that the functor $(\ \ )_r$ preserves monomorphisms, so the diagonal map $Y_r \to Y_r \times_{X_r} Y_r$ factors through

$$Y_r \cong U_r = supp^{-1}(U) \hookrightarrow Y_r \times_{X_r} Y_r.$$

It shows that the diagonal map $Y_r \to Y_r \times_{X_r} Y_r$ is an open immersion.

Since a map $g : V \to W$ of topological spaces is a local homeomorphism if and only if both g and the diagonal $V \to V \times_W V$ are open maps, the proposition follows. □

5.2 Real étale topos is spatial

We now prove that the real étale topos is spatial, i.e., equivalent to the category of sheaves on a topological space.

Theorem 5.2.1. *For any scheme X, the topos $Sh(X_{ret})$ and the topos $Sh(X_r)$ are naturally equivalent.*

We will follow Scheiderer's proof ([27]), which uses an auxiliary site. Let us discuss it now.

Let X be a scheme. Define $X_{aux} = (\mathcal{C}, aux)$ to be the following site:

- An object of $\mathcal{C}$ is a pair (U, W) with $U \in Et/X$ and W an open subset of U_r.
- An arrow $(U', W') \to (U, W)$ in $\mathcal{C}$ is a morphism $f : U' \to U$ of X-schemes such that $f_r(W') \subseteq W$. In particular, f is étale.
- Let $f : (U', W') \longrightarrow (U, W)$ and $g : (U'', W'') \longrightarrow (U, W)$ be two arrows, the pair $(U' \times_U U'', W' \times_W W'')$ is clearly the pullback of f and g

$$\begin{array}{ccc} (U' \times_U U'', W' \times_W W'') & \xrightarrow{q} & (U', W') \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ (U'', W'') & \xrightarrow[g]{} & (U, W). \end{array}$$

  Thus, the pullback always exists, and consequently, all finite inverse limits exist.
- A family $\{f_i : (U_i, W_i) \to (U, W)\}_{i \in I}$ of arrows in $\mathcal{C}$ is a covering of (U, W) if and only if $W = \bigcup_{i \in I} (f_i)_r(W_i)$.

Regard the space X_r as a site in the usual way. Both Et/X and $O(X_r)$ (the category of open subsets of X_r) are full subcategories of $\mathcal{C}$ in canonical way.

These inclusions define morphisms of sites

$$X_{ret} \overset{\varphi}{\leftarrow} X_{aux} \overset{\psi}{\rightarrow} X_r.$$

Lemma 5.2.2. *The morphism φ defines an equivalence of categories $Sh(X_{ret}) \cong Sh(X_{aux})$.*

Proof. Since φ transforms coverings into coverings and preserves fiber products, the Comparison Lemma (1.3.22) implies that the topos morphism $\varphi : Sh(X_{aux}) \to Sh(X_{ret})$ is an equivalence, provided that each object (U, W) of $\mathcal{C}$ can be covered by objects (V, V_r) with $V \in Et/U$. In other words, given $\xi \in U_r$ and an open neighborhood W of ξ in U_r, we have to find an étale morphism $f : V \to U$ with $\xi \in f_r(V_r) \subseteq W$. We can assume that U is an affine scheme $U = Spec\ A$, and that $W = U(a)$ is an open sub-base. Since W is non-empty, it follows that 2 is not zero-divisor in A. Consider the canonical morphisms

$$A \xrightarrow{\phi_1} B = (A[T]/(T^2 - a)A[T]) \xrightarrow{\phi_2} B_{2a},$$

since

$$\overline{(T^2 - a)'(T/2a)} = \overline{\frac{2T^2}{2a}} = \overline{\frac{2a}{2a}} = 1,$$

$\phi_2 \circ \phi_1$ is a standard étale homomorphism. Hence, we have an étale morphism of schemes f induced by $\phi_2 \circ \phi_1$. Since a is a square and a is a unit ($\frac{2}{2a}$ is a inverse of a), we have $f_r(Sper\ B_{2a}) \subseteq W$.

It remains to show that $\xi \in f_r(Sper\ B_{2a})$. Since $(A[T]/(T^2 - a)A[T])$ is a free A-module of rank 2, ϕ_1 is faithfully flat, i.e.,

$$Spec(\phi_1) : Spec\ B \to Spec\ A$$

is surjective. By the hypothesis, $\xi(2a) = 2\xi(a) > \xi(a) > 0$, in particular

$$supp\ \xi \in D_A(2a) = \{\mathfrak{p} \in Spec\ A : 2a \notin \mathfrak{p}\}.$$

Therefore, $Spec(\phi_1)^{-1}(supp\ \xi) \subseteq D_B(2a)$. But

$$Spec(\phi_2) : Spec\ B_{2a} \cong D_B(2a) \to Spec\ B$$

is an open immersion, so there exists $\mathfrak{q} \in Spec\ B_{2a}$ such that $f(\mathfrak{q}) = supp\ \xi$. It is now clear that there is an "extension" of ξ via f, which implies $\xi \in f_r(Sper\ B_a)$. □

It remains to show that the topos morphism induced by ψ is also an equivalence.

Let $S \in Sh(X_{aux})$. For every $U \in Et/X$ the map given by $W \mapsto S(U, W)$ (for $W \subseteq U_r$ open) defines a sheaf on U_r, which we denote by S_U. If $f : V \to U$ is a morphism in Et/X, the restriction maps of S define an f_r-morphism from S_U to S_V, i.e., a sheaf map $f_r^* S_U \Rightarrow S_V$ on V_r. To be more precise, for each open $W \subseteq V_r$ open, we have $f_r(W) \subseteq f_r(W)$. Hence, the morphism f is an arrow from (V, W) to $(U, f_r(W))$. Thus, there is a restriction map $S(f) : S(U, f_r(W)) \to S(V, W)$. To ensure that $f_r^* S_U \Rightarrow S_V$ is a sheaf map, we need to show that the following diagram commutes

$$\begin{array}{ccccc} W & \quad & S(U, f_r(W)) & \xrightarrow{S(f)} & S(V, W) \\ \uparrow & & \downarrow{\scriptstyle S(id_U)} & & \downarrow{\scriptstyle S(id_V)} \\ W' & & S(U, f_r(W')) & \xrightarrow[S(f)]{} & S(V, W'), \end{array}$$

but it is elementary

$$S(f) \circ S(id_U) = S(id_U \circ f) = S(f) = S(f \circ id_V) = S(id_V) \circ S(f).$$

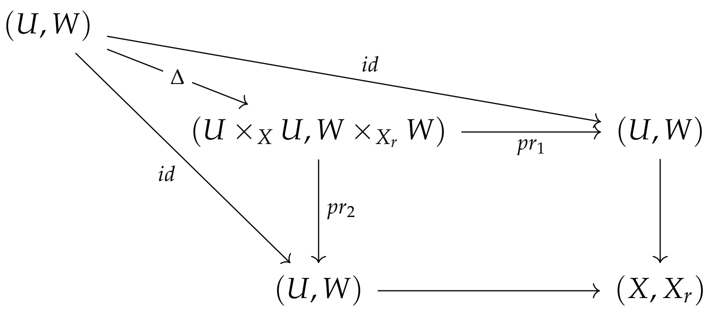
Lemma 5.2.3. *The sheaf map $f_r^* S_U \Rightarrow S_V$ is an isomorphism.*

Proof. It is enough to show that, for any $f : U \to X$ étale, the sheaf map $f_r^* S_X \Rightarrow S_U$ is an isomorphism: Let $g : V \to U$ be an étale morphism of schemes, since the map $f_r^* S_X \Rightarrow S_U$ and the map $g_r^* f_r^* S_X = (f \circ g)_r^* S_X \Rightarrow S_V$ are isomorphisms, $g_r^* S_U \to S_V$ is a sheaf isomorphism.

Since f_r is a local homeomorphism, there exists an open subset W of U_r such that $f_r|_W$ is injetive, i.e., an open immersion. Since $f : (U, W) \to (X, f_r(W))$ is a covering in $\mathcal{C}$, by the definition of sheaf, we have an exact sequence (of sets):

$$(f_r^* S_X)(W) = S_X(f_r(W)) \xrightarrow{\quad i \quad} S_U(W) \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} S_{U \times_X U}(W \times_{X_r} W)$$

with $W \times_{X_r} W \subseteq U_r \times_{X_r} U_r = (U \times_X U)_r$ (by Corollary 5.1.6). Since $f_r|_W$ is injective, the diagonal map $W \to W \times_{X_r} W$ is bijective. So the diagonal morphism $(U, W) \to (U \times_X U, W \times_{X_r} W)$ is a covering in $\mathcal{C}$, in particular $S_{U \times_X U}(W \times_{X_r} W) \to S(U, W)$ is injective (because it is an equalizer). But this implies that the two maps pr_1^*, pr_2^* in the sheaf condition coincide, since the pullback diagram



induces a sequence of morphisms

$$S(U,W) \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} S(U \times_X U, W \times_{X_r} W) \overset{S(\Delta)}{\hookrightarrow} S(U,W)$$

which means that $pr_1^* = pr_2^*$. Therefore, the sheaf map $(f_r^* S_X)(W) \to S_U(W)$ is bijective, since Ker $(pr_1^*, pr_2^*) = \text{Im } i$.

Since f_r is a local homeomorphism, U_r has a basis $\{W_i\}$ of open sets such that $f_r|_{W_i}$ is injective for every i. Since the sheaf map $f_r^* S_X \Rightarrow S_U$ is a bijection on every basic open, it is a global isomorphism. □

We now prove that $Sh(X_{ret})$ and $Sh(X_r)$ are equivalent. We denote the geometric morphism induced by ψ (resp. φ) by $(\psi^*, \psi_*) : Sh(X_{aux}) \to Sh(X_r)$ (resp. $(\varphi^*, \varphi_*) : Sh(X_{aux}) \to Sh(X_{ret})$).

$$Sh(X_{ret}) \underset{\varphi^*}{\overset{\varphi_*}{\leftrightarrows}} Sh(X_{aux}) \underset{\psi^*}{\overset{\psi_*}{\rightleftarrows}} Sh(X_r)$$

Proof. For $F \in Sh(X_r)$, the sheaf $\psi^* F$ on $Sh(X_{aux})$ is the sheaf associated with the presheaf $(U,W) \mapsto F(f_r(W))$, where $f : U \to X$ denotes the structural morphism of $U \in Et/X$. Using Proposition 5.1.7, we have

$$(\psi^* F)(U,W) = (f_r^* F)(W) \ \ i.e., \ \ (\psi^* F)_U = f_r^* F.$$

On the other hand, ψ_* associates $S \in Sh(X_{aux})$ to $S_X \in Sh(X_r)$. So Lemma 5.2.3 ensures that $\psi^* \psi_* S \to S$ is an isomorphism. Since the other adjunction map $F \to \psi_* \psi^* F$ is also an isomorphism, we have that $Sh(X_{aux})$ and $Sh(X_r)$ are equivalent. By the Lemma 5.2.2, the topos $Sh(X_{ret})$ and the topos $Sh(X_r)$ are naturally equivalent. □

Now, by glueing these morphisms, we obtain a topos maps $\#, \flat$ between $Sh(X_{ret})$ and $Sh(X_r)$

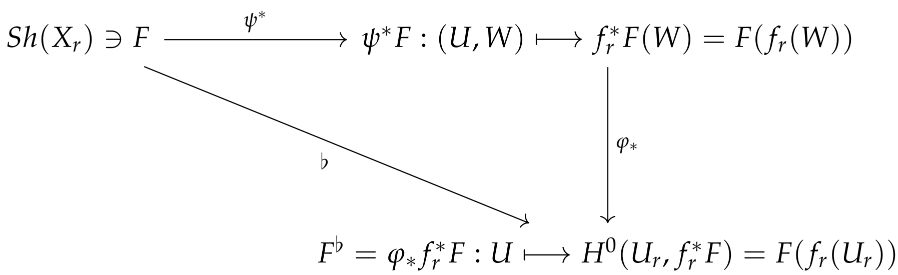
$$Sh(X_{ret}) \underset{\flat}{\overset{\#}{\rightleftarrows}} Sh(X_r)$$

given by

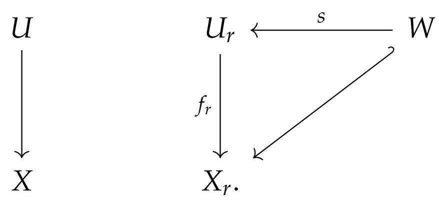
$$Sh(X_{ret}) \ni G \longmapsto G^\# := \psi_* \varphi^* G \in Sh(X_r);$$
$$Sh(X_r) \ni F \longmapsto F^\flat := \phi_* \psi^* F \in Sh(X_{ret}).$$

So the compositions $\flat \circ \#$ and $\# \circ \flat$ are naturally isomorphic to the identity functors. Which means

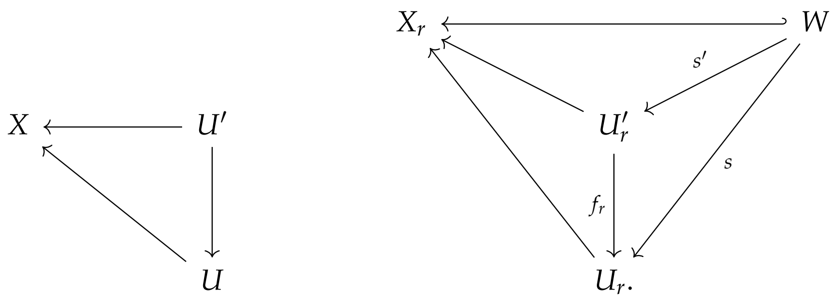
In detail, the map $\flat$ acts as shown in the commutative diagram:



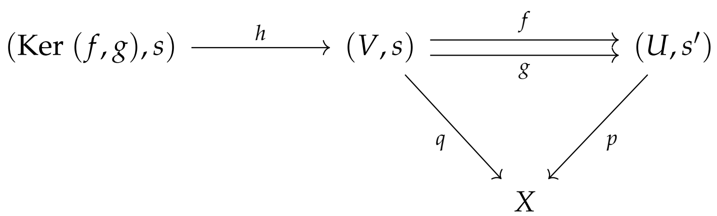
To make $G^\#$ more explicit, we need to introduce some additional notation. Fix an open subset W of X_r, and let I_W be the category of all pairs (U, s), where $U \in Et/X$ and $s : W \to U_r$ is a continuous section of $U_r \to X_r$ over W. This leads to the following commutative diagram:



And an arrow $(U', s') \to (U, s)$ in I_W is an X-morphism $f : U' \to U$ such that $s = f_r \circ s'$



The category I_W is a left filtering: Given a diagram



where p, q are étale morphisms such that $p \circ f = p \circ g = q$, and h is the kernel/equalizer of f and g. By the definition of a morphism in I_W, the section $s : W \to V_r$ satisfies $f_r \circ s = g_r \circ s$. Then h is étale, and $h_r : (\text{Ker}(f, g))_r \to V_r$ is the kernel of f_r and g_r by Proposition 5.1.6, thus s factors uniquely through h_r.

Let $W' \subseteq W$ be an inclusion of open subsets of X_r, there is a natural functor $I_W \to I_{W'}$ given by restriction. Hence if P is a presheaf on Et/X, then the functor $P^{\dagger} : X_r \to \mathbf{Set}$ given by

$$P^{\dagger}(W) = \varinjlim_{(U,s) \in I_W^{op}} P(U)$$

is a presheaf on the topological space X_r.

Now we are ready to describe the maps # and $\flat$.

Theorem 5.2.4. *1. If F is a sheaf on X_r, then $F^{\flat}$ is the sheaf on X_{ret} which sends $(U \xrightarrow{f} X) \in Et/X$ to $H^0(U_r, f_r^* F) = F(f_r(U_r))$.*

2. If P is a presheaf on Et/X and G is the associated sheaf respect to real étale topology ret, then $G^{\#}$ is the sheaf on X_r which is associated to the presheaf $P^{\flat}$. In diagrammatic form, this implies that the diagram

$$\begin{array}{ccc} Psh(Et/X) & \xrightarrow{\quad \dagger \quad} & Psh(X_r) \\ \downarrow{\scriptstyle a_{ret}} & & \downarrow{\scriptstyle a_r} \\ Sh(X_{ret}) & \xrightarrow{\quad \# \quad} & Sh(X_r) \end{array}$$

commutes.

Proof. (1) : We have already established.
(2) : Let x be the Zariski spectrum of a real closed field R, and let $\alpha : x \to X$ be a morphism of schemes, representing a point $\xi \in X_r$. Then the stalk of the presheaf $P^{\dagger}$ in ξ is

$$P_{\xi}^{\dagger} = \varinjlim_{\substack{W \subseteq X_r \text{ open} \\ \xi \in W}} P^{\dagger}(W) = \varinjlim_{x \underset{X}{\to} U} P(U)$$

where the second direct limit is taken over the category of all X-morphism from x into étale X-Scheme U.

Consider the natural morphism $P \to G = a_{ret}(P)$ of presheaves on Et/X. For any ξ as above the induced map $P_{\xi}^{\dagger} \to G_{\xi}^{\dagger}$ is bijective (the key point is that $Sh(X_{ret})$ is spatial). Hence the induced map $P^{\dagger} \to G^{\dagger}$ of presheaves on X_r (a topological space) becomes an isomorphism if we applying the sheafification functor, since it is bijective on stalks.

This shows that it suffices to prove the case when $P = G$ is a sheaf on X_{ret}. Since # and $\flat$ are known to be quasi-inverses of each other, it is enough to prove for any sheaf F on X_r, F is isomorphic to the sheaf associated to $(F^\flat)^\dagger$, in other words, the diagram

$$\begin{array}{ccc} PSh(X_{ret}) & \xrightarrow{\ \dagger\ } & PSh(X_r) \\ \uparrow & & \downarrow{\scriptstyle a_r} \\ Sh(X_{ret}) & \underset{\flat}{\overset{\#}{\rightleftarrows}} & Sh(X_r) \end{array}$$

is commutative.

Let $G := F^\flat$ be the image of F by $\flat$. Let $W \subseteq X_r$ be an open subset, and let $(U, s) \in I_W$. Since the diagram

$$\begin{array}{ccc} U_r & \xrightarrow{\ f_r\ } & X_r \\ {\scriptstyle s}\uparrow & \nearrow & \\ W & & \end{array}$$

commutes and f_r is open (By Lemma 5.1.4), we have $f_r(U_r) \supseteq W = f_r \circ s(W)$ is an open subset. Therefore, there is a natural map $s^* : G(U) = (f_r^* F)(U_r) \to F(W) = (s^* f_r^* F)(W) = F(f_r \circ s(W))$ induced by restriction, namely the pullback by s.

These maps fit together to give a morphism $G^\flat \to F$ of presheaves on X_r. But the stalk maps $G^\dagger_\xi \to F_\xi$ are bijective

$$\varinjlim_{\substack{W \subseteq X_r \text{ open} \\ \xi \in W}} G^\dagger(W) = \varinjlim_{\substack{W \subseteq X_r \text{ open} \\ \xi \in W}} \varinjlim_{(U,s) \in I_W} F(f_r(U_r)) = \varinjlim_{\substack{W \subseteq X_r \text{ open} \\ \xi \in W}} F(W).$$

Therefore $F \cong a_r(G^\dagger)$ as desired. □

The composite map $a_r \circ \#$ generalizes the construction introduced by Coste and Roy in [26] to define the abstract Nash sheaf on the real spectrum. If we restrict this construction to étale A-algebras, and consider the étale structure sheaf, i.e., $\mathcal{O}_X(B) = B$ for every étale B-algebra, then $a_r \circ \#(\mathcal{O}_X)$ exactly gives the abstract Nash sheaf defined in [26]. This shows that the Nash sheaf is the natural "structure" sheaf for the real spectrum. Moreover, when $A = \mathbb{R}[x_1, ..., x_n]$, the abstract Nash sheaf corresponds (though not identically or isomorphically) to the classical Nash sheaf, whose sections are collections of algebraic and analytic functions. This correspondence follows from Artin-Mazur's description of the Nash sheaf. Therefore, the Nash sheaf is a fundamental object in the study of real algebraic geometry.

Chapter 6

Glueing sites

In this chapter, we investigate the "glueing" of the étale topos and the real étale topos, a concept introduced by Claus Scheiderer in his work on real and étale cohomology ([27]). In the first chapter, we construct the b-topology and show that this category can be viewed, in practice, as a space obtained by glueing the categories of sheaves $Sh(X_{et})$ and $Sh(X_{ret})$. As a result, we demonstrate that the canonical functors between these topoi possess favorable properties in second section.

6.1 A glued space

The result from chapter before allows us to replace X_r by X_{ret}, a site modeled on the category Et/X as X_{et}. A natural question arises: is there a relationship between X_{et} and X_{ret} (or X_r)?

The two topologies X_{et} and X_{ret} cannot be directly compared because neither is finer nor coarser than the other. However, we can attempt a comparison via an intermediate topology: the intersection of the two topologies, i.e., the finest common coarsening of b of et and ret. We denote the site $(Et/X, b)$ by X_b.

Intuitively, the topos $Sh(X_b)$ contains both $Sh(X_{et})$ and $Sh(X_{ret})$. In this section, we will show that $Sh(X_{et})$ is an open subtopos of $Sh(X_b)$, while $Sh(X_{ret})$ is its closed complement. As a consequence, X_b can be understood as the result of glueing $Sh(X_{ret})$ to $Sh(X_{et})$.

Definition 6.1.1. *Let X be a scheme. The topology on Et/X which is the intersection of the étale and the real étale topology, denoted by b. Thus, a family $\{U_i \to U\}_{i \in I}$ in Et/X is a covering for the topology b if and only if it is both surjective and real surjective. We denote the site $(Et/X, b)$ by X_b. Moreover, we write*

$$j = (j^*, j_*) : Sh(X_{et}) \to Sh(X_b)$$

and

$$i = (i^*, i_*) : Sh(X_{ret}) \to Sh(X_b)$$

for the canonical topos embeddings.

Remark 6.1.2. *Since the morphisms j_* and i_* are embeddings, their left adjoints j^*, i^* are shefification functors.*

Let us verify that Proposition 1.3.32 applies. For Y a scheme and p a prime number, let $Y_{(p)}$ be the largest open subscheme of Y on which p is invertible. If ξ_{p^n} is a primitive p^n-th root of unity, then $Y_{(p)}[\xi_{p^n}] := Y_{(p)} \times_{spec\,\mathbb{Z}} spec\ \mathbb{Z}[\xi_{p^n}] \xrightarrow{pr_1} Y_{(p)}$ is an étale covering (since, in the affine case, $Y_{(p)}[\xi_{p^n}]$ is a finite free $Y_{(p)}$-module, therefore pr_1 is faithful flat), where $\mathbb{Z}[\xi_{p^n}]$ denotes the group ring generated by ξ_{p^n}.

Consider two morphisms obtained by composing with the inclusion map

$$Y_{(2)}[\sqrt{-1}] \xrightarrow{pr_1} Y$$

and

$$Y_{(3)}[\xi_3] \xrightarrow{pr_1} Y.$$

Since $\sqrt{-1}$ is a sum of squares in the residue fields of $Y_{(2)}[\sqrt{-1}]$ and $Y_{(3)}[\xi_3]$, both schemes have empty real spectrum, therefore empty sieve is a real étale covering of $Y_{(2)}[\sqrt{-1}]$ and $Y_{(3)}[\xi_3]$. We need to show that these two morphisms form an étale covering. Suppose $Y = Spec\ A$ is affine, by the definition of $Y_{(p)}$, we have

$$Y_{(2)} \supseteq D(2) = \{\mathfrak{p} \in Y : 2 \notin \mathfrak{p}\}$$

and $Y_{(3)} \supseteq D(3)$. Therefore, $Y = Y_{(2)} \cup Y_{(3)}$ (if $\mathfrak{p} \in Y$ does not belong to both subschemes, $2, 3 \in \mathfrak{p}$ and so $1 \in \mathfrak{p}$, this is a contradiction). Hence, $Y_{(2)}[\sqrt{-1}]$ and $Y_{(3)}[\xi_3]$ cover Y in the étale topology.

By Proposition 1.3.32 we therefore get,

Proposition 6.1.3. *The morphism $j : Sh(X_{et}) \to Sh(X_b)$ is an open topos embedding, and $i : Sh(X_{ret}) \to Sh(X_b)$ is the embedding of the closed complement.*

Now, we will define the glueing functor such that relates $Sh(X_{et})$ and $Sh(X_{ret})$.

Definition 6.1.4. *The **glueing functor** is the functor $\rho := i^* j_* : Sh(X_{et}) \to Sh(X_{ret})$.*

Proposition 6.1.5. *The glueing functor is left exact, i.e., preserves finite inverse limits.*

Proof. This result holds for any open subtopos and its closed complement (see [SGA4.IV.9.5.4] [12]). In particular, it applies to the real étale topos and the étale topos by Proposition 6.1.3. $\square$

Consider triples (B, A, ϕ), where $B \in Sh(X_{ret})$, $A \in Sh(X_{et})$ and $\phi : B \to \rho A$ is a morphism of sheaves on X_{ret}. If (B, A, ϕ) and (B', A', ϕ') are such triples, a morphism between (B, A, ϕ) and (B', A', ϕ') is a pair (m, m'), where $m \in Sh(X_{ret})(B, B')$ and $m' \in Sh(X_{et})(A, A')$ such that the following diagram

$$\begin{array}{ccc} B & \xrightarrow{\phi} & \rho(A) \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \rho(m')} \\ B' & \xrightarrow{\phi'} & \rho(A') \end{array}$$

commutes. These triples with morphisms as defined above form a category which is denoted $(Sh(X_{ret}), Sh(X_{et}), \rho)$. For more general construction of such category, see [12].

We have a natural functor

$$In : Sh(X_b) \to (Sh(X_{ret}), Sh(X_{et}), \rho)$$

given by

$$\begin{array}{ccc} F & & (i^*F, j^*F, \phi_F : i^*F \to i^*j_*j^*F = \rho j^*F) \\ \downarrow{\scriptstyle n} & \longmapsto & \downarrow{\scriptstyle (i^*n, j^*n)} \\ F' & & (i^*F', j^*F', \phi_{F'} : i^*F' \to \rho j^*F') \end{array}$$

Here ϕ_F comes from the adjunction map $F \xrightarrow{adj} j_*j^*F$ via functor i^*, i.e., $\phi_F = i^*(adj)$.

By [12][SGA.IV.9.5], the functor In is an equivalence.

Proposition 6.1.6. *The functor In is an equivalence of categories between $Sh(X_b)$ and $(Sh(X_{ret}), Sh(X_{et}), \rho)$. A quasi-inverse is given by the functor which sends a triple $(B, A, \phi : B \to \rho A)$ to the Pullback in $Sh(X_b)$*

$$\begin{array}{ccc} i_*B \times_{i_*\rho A} j_*A & \longrightarrow & j_*A \\ \downarrow & & \downarrow{\scriptstyle adj} \\ i_*B & \xrightarrow[i_*B]{} & i_*\rho A \end{array}$$

This result holds for any open subtopos and its closed complement.

This proposition guarantees that $Sh(X_b)$ is a space constructed by glueing the categories of sheaves $Sh(X_{et})$ and $Sh(X_{ret})$. This facilitates the study of the canonical morphisms between these toposes.

6.2 Some useful morphism

Identifying $Sh(X_b)$ with $(Sh(X_{ret}), Sh(X_{et}), \rho)$ via the above equivalence, we obtain

$$\begin{aligned} j_* &: Sh(X_{et}) \to Sh(X_b) & A &\mapsto (\rho A, A, id : \rho A \to \rho A) \\ j^* &: Sh(X_b) \to Sh(X_{et}) & (B, A, \phi) &\mapsto A \\ i_* &: Sh(X_{ret}) \to Sh(X_b) & B &\mapsto (B, *, B \to *) \\ i^* &: Sh(X_b) \to Sh(X_{ret}) & (B, A, \phi) &\mapsto B. \end{aligned}$$

Corollary 6.2.1. *The functor $j^* : Sh(X_b) \to Sh(X_{et})$ has a left adjoint*

$$\begin{aligned} j_! : Sh(X_{et}) &\longrightarrow Sh(X_b) \\ A &\longmapsto (\emptyset, A, \emptyset \to \rho A) \end{aligned}$$

where $\emptyset$ is the initial sheaf.

Proof. For every $F \in Sh(X_{et})$ and $G = (B, A, \phi) \in (Sh(X_{ret}), Sh(X_{et}), \rho) \cong Sh(X_b)$, a morphism (m, m') from $j_!F$ to G satisfies

$$\begin{array}{ccc} \emptyset & \longrightarrow & \rho F \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \rho(m')} \\ B & \xrightarrow[\phi]{} & \rho A. \end{array}$$

Since $\emptyset$ is an initial object, the morphism (m, m') is uniquely determined by m'. Hence,

$$hom(j_!F, G) \cong hom(F, A) \cong hom(F, j^*G).$$

□

We have the same result for the sheaves of abelian groups:

Corollary 6.2.2. *1. The functor $j^* : Ab(X_b) \to Ab(X_{et})$ has an exact additive left adoint*

$$\begin{aligned} j_! : Ab(X_{et}) &\to Ab(X_b) \\ A &\mapsto (0, A, 0 \to \rho A), \end{aligned}$$

*called **"extension by zero"**, where 0 is the zero sheaf. In particular, j^* is an exact additive functor.*

2. *The functor $i_* : Ab(X_{ret}) \to Ab(X_b)$ has a right adjoint*

$$\begin{aligned} i^! : Ab(X_b) &\to Ab(X_{ret}) \\ (B, A, \phi : B \to \rho A) &\mapsto \text{Ker } \phi. \end{aligned}$$

In particular, the additive functor i_* is exact.

Proof. (1): The same reasoning as in the previous corollary applies here. The functor $j_!$ is clearly an exact additive functor.

(2): For every $F \in Sh(X_{ret})$ and $G = (B, A, \phi) \in (Sh(X_{ret}), Sh(X_{et}), \rho) \cong Sh(X_b)$, a morphism (m, m') from i_*F to G satisfies

$$\begin{array}{ccc} F & \longrightarrow & \rho 0 = 0 \\ {\scriptstyle m}\downarrow & \overset{0}{\searrow} & \downarrow{\scriptstyle \rho(m')} \\ B & \xrightarrow[\phi]{} & \rho A \end{array}$$

Hence, $\phi \circ m = 0 = 0 \circ m$. By the universal property of the kernel,

$$\begin{array}{ccccc} F & & & & \\ \vdots & \overset{m}{\searrow} & & & \\ \text{Ker } m & \longrightarrow & B & \overset{\phi}{\underset{0}{\rightrightarrows}} & \rho A \end{array}$$

there exists a unique morphism from F to Ker m. Therefore, we have

$$hom(F, \text{Ker } \phi) = hom(i_*F, G)$$

as desired. □

The two functors $j_!$ (for set-valued and abelian sheaves, respectively) do not coincide on abelian sheaves.

Corollary 6.2.3. *For every $F \in Ab(X_b)$ there are natural exact sequences (of adjunction map) on $Ab(X_b)$*

$$0 \to j_!j^*F \to F \to i_*i^*F \to 0$$

and

$$0 \to i_*i^!F \to F \to j_*j^*F.$$

In particular, for $A \in Ab(X_{et})$ there is a natural exact sequence on $Ab(X_b)$

$$0 \to j_! A \to j_* A \to i_* \rho A \to 0.$$

Proof. Identifying the sheaf F with (B, A, ϕ), these sequences are equivalent to

$$0 \to (0, A, 0 \to \rho A) \to (B, A, \phi) \to (B, 0, B \to 0) \to 0$$

and

$$0 \to (\text{Ker } \phi, 0, \text{Ker } \phi \to 0) \to (B, A, \phi) \to (\rho A, A, id).$$

The verification of exactness of these sequences is trivial. □

Corollary 6.2.4. *Let X be a scheme, for every object $U \in Et/X$, $\epsilon_{et}(U) = \epsilon_b(U) = h_U$, while $\epsilon_{ret}(U) = \rho h_U$.*

Proof. Since h_U is an étale sheaf, we have $\epsilon_{et}(U) = h_U$, and therefore $\epsilon_b(U) = h_U$. Identifying the h_U with $j_*(h_U) = (\rho h_U, h_U, id)$, the *ret*-sheafification of h_U is

$$\epsilon_{ret}(U) = a_{ret}(h_U) = i^*(\rho h_U, h_U, id) = \rho h_U.$$

□

Example 6.2.5. *The b- topology coincides with the étale topology if and only if $X_r = \varnothing$. Thus the most basic proper example for the b-topology arises from $X = Spec\ R$ with R a real closed field.*

Let $G = Gal(R(\sqrt{-1})/R)$, which is isomorphic to $\mathbb{Z}_2$. Then $Sh(X_b)$ is equivalent to the category of all triples $(B, A, \phi : B \to A^G)$, where

- *B is a set,*
- *A a continuous G-set,*
- *ϕ is a map,*
- *and A^G denotes the elements of A fixed by G.*

*This equivalence follows from the correspondence between $Sh((Spec\ \ k)_{et})$ and $\mathcal{C}G$-**Set**, which will be established in the next chapter.*

Notation 6.2.6. *If $t \in \{et, b, ret\}$ and M is a set, denote by $\underline{M}_t$ the constant sheaf on X_t with value in M.*

Proposition 6.2.7. *j_* and ρ preserve constant sheaves. That is: if M is a set, then $j_*\underline{M}_{et} = \underline{M}_b$ and $\rho\underline{M}_{et} = \underline{M}_{ret}$.*

Proof. $\underline{M}_t$ is the coproduct in $Sh(X_t)$ of M copies of the constant sheaf $*$. Since j_* is an "inclusion", there is a canonical sheaf isomorphism,

$$\underline{M}_b = \coprod_M j_*(*) = \coprod_M * \longrightarrow j_*(\coprod_M *) = j_*(\underline{M}_{et}).$$

After applying i^*, we obtain a sheaf morphism

$$\underline{M}_{ret} \longrightarrow \rho\underline{M}_{et}.$$

We need to verify that the morphism above is an isomorphism. However, this follows immediately, since the induced fiber maps are isomorphisms, and X_{ret} is spatial. □

Chapter 7

Applications

In the last chapter, we discuss the relationship between Galois cohomology and étale cohomology on a spectrum of fields, as well as the interplay among real points, orderings, and cohomological dimensions. The primary references for this chapter are [27] and [32].

In the first section, we prove that the category $Spec\ (k_{et})$ is equivalent to the category $\mathcal{C}G$-**Set**, and that the category $Ab((Spec\ k)_{et})$ is equivalent to the category $\mathcal{C}G$-**Mod**. As a corollary, we deduce that the étale cohomology of a field k coincides with the Galois cohomology of the absolute Galois group $Gal(k^{sep}/k)$.

In the second section, we discuss a bit about the connection between order and cohomological dimension, especially in the context of 2-torsion sheaves.

7.1 Galois and Étale cohomology

Let k be a field. Let k^{sep} be the separable closure of k and let G denote the Galois group of $Gal(k^{sep}/k)$ equipped with the canonical structure of a profinite group.

For each k-scheme X, we denote by $X(k^{sep})$ the set of k-morphism $Spec\ k^{sep} \to X$, called the set of k^{sep}-valued points on X. A k^{sep}-valued point of X corresponds uniquely to a point $x \in X$ together with a k-homomorphism $\kappa(x) \to k^{sep}$.

The group G acts from the left on $X(k^{sep})$: Let $g \in G$, we define the action of g on a point $Spec\ k^{sep} \to X$ by composing it from the left with the induced morphism $Spec(g) : Spec\ k^{sep} \to Spec\ k^{sep}$. By the Fundamental Theorem of Galois Theory if H is an open subgroup of G, then we can identify the set of fixed points $X(k^{sep})^H$ with the set $X(k')$ of all k'-valued points on X. Here k' is the fixed field of H, and the inclusion $X(k') \subseteq X(k^{sep})$ is induced by the canonical morphism $Spec\ k^{sep} \to Spec\ k'$. Since $X(k^{sep}) = \bigcup_H X(k^{sep})^H$, G acts continuously on $X(k^{sep})$.

Let T_G denote the canonical topology on the category $\mathcal{C}G$-**Set**. We have

Theorem 7.1.1. *The functor*

$$f : Et/k \longrightarrow \mathcal{C}G\text{-}\textbf{Set}$$
$$X \longmapsto X(k^{sep})$$

*is an equivalence of topologies between the étale site $\mathcal{C}G$-**Set** equipped with the canonical topology T_G and $Spec(k)_{et}$, i.e., the functor f is an equivalence of underlying categories, and that both f and any functor quasi-inverse to f are morphisms of topologies.*

Proof. First note that $(X \times_Z Y)(k^{sep}) \cong X(k^{sep}) \times_{Z(k^{sep})} Y(k^{sep})$, hence f commutes with fiber products.

Let $\{U_i \to U\}$ be a family of étale morphism of k-schemes. We want to show that $\{U_i \to U\}$ is a covering in $Spec(k)_{et}$ if and only if $\{U_i(k^{sep}) \to U(k^{sep})\}$ is a covering in T_G. Since both categories have arbitrary (direct) sums/coproducts (By Corollary 2.1.19) and since f commutes with sums, it is sufficient to show that a morphism $Y \to X$ of étale k-schemes is surjective if and only if $Y(k^{sep}) \to X(k^{sep})$ is surjective.

- Assume that $Y \to X$ is surjective. Let $x \in X$ and let a k-homomorphism $\kappa(x) \to k^{sep}$ be given. If $y \in Y$ lies above x, by the definition of étale morphism the extension $\kappa(y)/\kappa(x)$ is finite and separable, and therefore $\kappa(x) \to k^{sep}$ extends to a k-homomorphism $\kappa(y) \to k^{sep}$. But this means that $y(k^{sep}) \mapsto x(k^{sep})$ is surjective.

- Now, assume that $y(k^{sep}) \mapsto x(k^{sep})$ is surjective. Let $x \in X$. Since $\kappa(x)/k$ is finite and separable, there is a k^{sep}-valued point corresponding to x. If we take any k^{sep}-valued point of Y lying above it, then the corresponding point $y \in Y$ lies above x. Hence $Y \to X$ is onto.

It remains to show that f is an equivalence of categories. To prove this we first show the existence of the left adjoint functor ${}^{ad}f$ of f, and then check that the adjoint morphisms are isomorphisms..

To show the existence of ${}^{ad}f$ it is enough to check that the functor

$$X \longmapsto hom_G(U, X(k^{sep}))$$

is representable for all continuous G-sets U.

Now, each continuous G-set is equal to the direct sum of its orbits, and each orbit is isomorphic to a continuous G-set of the form G/H for an open subgroup H of G (by orbit-stabilizer theorem).

So it is sufficient to show that the functors

$$X \mapsto hom_G(G/H, X(k^{sep}))$$

are representable, since the category of étale k-schemes has arbitrary coproducts.

Let k' be the fixed field of the open subgroup H. Then $Spec\ k'$ is an étale k-scheme, and we have the isomorphisms

$$hom_G(G/H, X(k^{sep})) \cong X(k^{sep})^H \cong X(k') = hom_k(Spec\ k', X),$$

which are functorial in X. Hence $X \mapsto hom_G(G/H, X(k^{sep}))$ is represented by the object $Spec\ k'$.

The adjoint map $G/H \to f({}^{ad}f(G/H)) = Spec(k')(k^{sep})$ is a G-map, which sends the class $e \cdot H$ to the k^{sep}-valued point $Spec\ k^{sep} \to Spec\ k'$ corresponding to the inclusion $k' \subseteq k^{sep}$. But this map is an isomorphism. Since f and ${}^{ad}f$ commute with the direct sums/coproducts, we obtain $id \cong f \circ {}^{ad}f$. Analogue, we obtain $id \cong {}^{ad}f \circ f$ which completes the proof of the theorem. □

Corollary 7.1.2. *Let k be a field, and let k^{sep} be the separable closure of k. Let $G = Gal(k^{sep}/k)$ be the topological group equipped with profinite topology. Then we have*

$$Sh((Spec\ k)_{et}) \cong \mathcal{C}G\text{-}\textbf{Set}$$

Proof. By Proposition 1.4.18 and the Theorem above, for any sheaf $F \in Sh((Spec\ k)_{et})$, the map

$$F \longmapsto F \circ {}^{ad}f \longmapsto \varinjlim_H F({}^{ad}f(G/H)) = \varinjlim_{k'} F(Spec\ k')$$

defines an equivalence of categories between $Sh((Spec\ k)_{et})$ and $\mathcal{C}G$-**Set**. □

By Proposition 1.4.18 and the Corollary above, the map

$$F \longmapsto \varinjlim_{k'} F(Spec\ k') \longmapsto hom_G(-, \varinjlim_{k'} F(Spec\ k'))$$

defines an equivalence of categories between $Sh((Spec\ k)_{et})$ and $Sh(\mathcal{C}G\text{-}\textbf{Set}, T_G)$. Since $Spec\ k$ corresponds to $(Spec\ k)(k^{sep}) = \{e\}$, we obtain

$$\Gamma(Spec\ k, F) \cong \Gamma(e, hom_G(-, \varinjlim_{k'} F(Spec\ k'))) \cong (\varinjlim_{k'} F(Spec\ k'))^G.$$

Therefore, we have:

Corollary 7.1.3. *Let k be a field, and let k^{sep} be the separable closure of k. Let $G = Gal(k^{sep}/k)$ be a topological group equipped with profinite topology. Then we have*

1. *The functor*

$$\begin{aligned} Ab((Spec\ k)_{et}) &\longrightarrow \mathcal{C}G\text{-}\boldsymbol{Mod} \\ F &\longmapsto \varinjlim_{k'} F(Spec\ k') \end{aligned}$$

is an equivalence between the category of abelian sheaves on $Spec(k)_{et}$ and the category of continuous G-modules. Here k' runs through all finite (or only through all finite normal) extensions of k in k^{sep}.

2. *For every abelian sheaves F on $Spec(k)_{et}$ we have ∂-functorial isomorphisms*

$$H^i_{et}(Spec\ k, F) \cong H^i(G, \varinjlim_{k'} F(Spec\ k')),$$

where right-hand side denotes Galois-cohomology.

One can check that $\varinjlim_{k'} F(Spec\ k')$ is exactly the stalk $F_{Spec\ k^{sep}} = \varinjlim_{U,u} F(U)$ of F at the point $Spec\ k$, where the limit is over the affine étale neighborhoods (U, u) of $Spec\ k^{sep}$ (See [21]).

Corollary 7.1.4. *Let k be the separably closed. Then the functor $F \mapsto F(Spec\ k)$ is an equivalence between the category of abelian sheaves on $Spec(k)_{et}$ and the category **Ab**. So for all sheaves F on $Spec(k)_{et}$ we have*

$$H^i_{et}(Spec\ k, F) = 0$$

for $i > 0$.

Remark 7.1.5. *Let T denote the set of involutions in G, i.e., set of elements of order 2. This set is a closed subset of G, and acts by conjugation. By the Artin–Schreier theorem, the quotient topological space T/G is the real spectrum $Sper\ k$.*

*Upon identifying Et/k with $\mathcal{C}G$-**Set**, the real étale topology ret on $\mathcal{C}G$-**Set** is defined as follows: a family $\{U_i \to U\}_{i \in I}$ is a covering if and only if $\{U_i^t \to U^t\}_{i \in I}$ is a surjective family for every $t \in T$, where U^t denotes the t-invariant of U.*

Therefore, a $\{U_i \to U\}_{i \in I}$ is a covering in the b-topology if and only if $\{U_i^t \to U^t\}_{i \in I}$ is a surjective family for every $t \in T \bigcup \{id\}$.

For further details, see Section 9 of [27].

7.2 Ordering and étale cohomology

In the this section, we discuss a bit about the connection between order and cohomological dimension, especially in the context of 2-torsion sheaves.

Given a ring A, if we want to "eliminate" the ordering of the residue fields of A, a natural approach is to add the square root of -1 to A. This construction can be realized by taking the tensor product. Consider the tensor product $A \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{-1}]$ of A and $\mathbb{Z}[\sqrt{-1}]$ over $\mathbb{Z}$. An element of the ring $A \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{-1}]$ is of the form $c \otimes_{\mathbb{Z}} a + bi$. By the definition of the tensor product, we have:

$$c \otimes_{\mathbb{Z}} a + bi = c \otimes_{\mathbb{Z}} a + c \otimes_{\mathbb{Z}} bi = ac \otimes_{\mathbb{Z}} 1 + bc \otimes_{\mathbb{Z}} i.$$

Thus, every element of $A \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{-1}]$ can be written as $x \otimes_{\mathbb{Z}} 1 + y \otimes_{\mathbb{Z}} i$, where $x, y \in A$. In other words, every element of $A \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{-1}]$ decomposes into a real part and an imaginary part, similar to a complex number.

This idea also extends to arbitrary schemes. Let X be a scheme, and define

$$X' := X \times_{Spec\ \mathbb{Z}} Spec\ \mathbb{Z}[\sqrt{-1}]$$

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow \\ Spec\ \mathbb{Z}[\sqrt{-1}] & \longrightarrow & Spec\ \mathbb{Z} \end{array}$$

(The map π is independent of the choice of the rightward arrow). Since the morphism $Spec\ \mathbb{Z}[\sqrt{-1}] \to Spec\ \mathbb{Z}$ is étale, the morphism π is also étale. Therefore, π induces the geometric morphism. We will write $\pi = (\pi^*, \pi_*) : Sh(X'_{et}) \to Sh(X_{et})$, instead of $\pi_{et} = (\pi^*_{et}, \pi_{et\,*})$. Since π is finite, by Proposition 2.3.3, the direct image functor π_* is exact.

Definition 7.2.1. *Let $X = (\mathcal{C}, \tau)$ be a site, p a prime number, and $F \in Ab(X)$ a sheaf.*

1. *We say that F is p-**primary torsion** if for every object $U \in \mathcal{C}$, the section $\Gamma(U, F)$ is a p-torsion abelian group, i.e., for every $x \in F(U)$, x has order p^n, for some $n \in \mathbb{N}$.*

2. *We say that F is **torsion** if for every object, the section $\Gamma(U, F)$ is a torsion abelian group.*

Definition 7.2.2. *Let $X = (\mathcal{C}, \tau)$ be a site, and let p be a prime number.*

1. *The **cohomological** p-**dimension** $cd_p(X)$ of X is defined as the largest integer n for which there exists a p-primary torsion sheaf F on X such that $H^n(X, F) \neq 0$. If no such integer exists, we write $cd_p(X) = \infty$.*

2. *The* ***cohomological dimension*** $cd(X)$ *of* X *is defined as the largest integer* n *for which there exists an abelian sheaf* F *such that* $H^n(X, F) \neq 0$.

Proposition 7.2.3. *Let* X *be a scheme and* p *a prime number. The following equalities hold:*

- $H^n_{ret}(X, A) = H^n_b(X, i_* A)$;
- $H^n_{et}(X', A) = H^n_{et}(X, \pi_* A)$

Proof. Since i^* and π^* are exact, by Lemma 1.2.35, i_* and π_* preserve the injective objects.

Since i_* is exact and represents an "inclusion" functor, by Proposition 1.2.36, we have

$$H^n_{ret}(X, A) = R^n\Gamma(A) = R^n(\Gamma \circ i_*)(A) = R^n\Gamma(i_* A) = H^n_b(X, i_* A)$$

for every $A \in Ab(X_{ret})$ and $n \geq 0$.

Note that

$$\Gamma(X', A) = A(X') = A(X \times_X X') = \pi_* A(X) = \Gamma(X, \pi_* A)$$

for every $A \in Ab(X'_{et})$. Then the section functor $R^n\Gamma_{X'}$ coincides with $R^n\Gamma_X \circ \pi_*$. Since π_* is exact, by the Proposition 1.2.36 again, we obtain

$$H^n_{et}(X', A) = R^n\Gamma_{X'}(A) = R^n\Gamma_X \circ \pi_*(A) = R^n\Gamma_X(\pi_* A) = H^n_{et}(X, \pi_* A).$$

□

Corollary 7.2.4. *Let* X *be a scheme and* p *a prime number. The following inequalities hold:*

$$cd_p(X_r) \leq cd_p(X_b), \quad cd_p(X'_{et}) \leq cd_p(X_{et}), \quad cd(X_r) \leq cd(X_b), \quad cd(X'_{et}) \leq cd(X_{et}).$$

Proof. Note that i_* and π_* maps p-primary torsion sheaves to p-primary torsion sheaves. By the previous proposition, if there exists an abelian sheaf (resp. p-primary torsion abelian sheaf) $A \in Ab(X_{ret})$ such that $H^n_{ret}(X, A) \neq 0$, then there is also an abelian sheaf (resp. p-primary torsion abelian sheaf) $i_* A \in Ab(X_b)$ such that $H^n_b(X, i_* A) \neq 0$. In particular, the inequalities $cd_p(X_r) \leq cd_p(X_b)$ and $cd(X_r) \leq cd(X_b)$ hold.

By a similar argument, the inequalities $cd_p(X'_{et}) \leq cd_p(X_{et})$ and $cd(X'_{et}) \leq cd(X_{et})$ hold. □

Now we state the theorem that relates $cd_p(X_r)$ and $cd_p(X'_{et})$.

Theorem 7.2.5. *If* X *is a scheme such that* 2 *is invertible in* $\mathcal{O}(X)$, *then* $cd_p(X_r) \leq cd_p(X'_{et})$ *for all prime numbers* p.

Proof. The proof of the theorem is quite technical; for this reason, we will omit it. The reader can find the proof in Sections 4, 5, and 7 (Part I) of [27]. □

Combining the theorem and the previous proposition, we obtain the following corollary.

Corollary 7.2.6. *If X is a scheme such that 2 is invertible in $\mathcal{O}(X)$ then $cd_p(X_r) \leq cd_p(X_{et})$ for all prime numbers p.*

This corollary can be interpreted as the ordering is an information that can be extracted from the étale cohomology.

We provide further discussion on it.

Proposition 7.2.7. *If k is a real closed field, then $H^1_{et}(Spec\ k, \mu_{2,Spec\ k}) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. By Corollary 7.1.3, it is enough to show that $H^1(G, \mu_2) \cong \mathbb{Z}/2\mathbb{Z}$, where $G = Gal(k[\sqrt{-1}]/k) = Gal(k^{sep}/k)$. It is clear that 2 is invertible in k and $k[\sqrt{-1}]$, so the Kummer sequence

$$1 \to \mu_2 \to k[\sqrt{-1}]^\times \xrightarrow{2} (k[\sqrt{-1}]^\times)^2 = k[\sqrt{-1}]^\times \to 1$$

is exact. Applying the functor $H^i(G, -)$, we obtain the long exact sequence:

$$1 \to \mu_2 \to k^\times \xrightarrow{2} k^\times \to H^1(G, \mu_2) \to H^1(G, k[\sqrt{-1}]^\times) \to ....$$

From Hilbert's Theorem 90, we know that $H^1(G, k[\sqrt{-1}]^\times) = 0$, so the map $k^\times \to H^1(G, \mu_2)$ is surjective. By the isomorphism theorem, we obtain

$$k^\times/(k^\times)^2 \cong H^1(G, \mu_2).$$

Finally, since any element of a real closed field can be written as $(-1)^n x^2$, we conclude that

$$H^1_{et}(Spec\ k, \mu_{2,Spec\ k}) \cong H^1(G, \mu_2) \cong k^\times/(k^\times)^2 \cong \mathbb{Z}/2\mathbb{Z}.$$

□

Proposition 7.2.8. *If k is a real closed field, then $cd_2(Spec(k)_{et}) = +\infty$.*

Proof. Let $k' = k[\sqrt{-1}]$. Since k is a real closed field, k' is algebraically closed, and in particular, is separably closed. By Corollary 7.1.4, for all sheaves F on $Spec(k)_{et}$ we have

$$H^i_{et}(Spec\ k, F) = 0$$

for $i > 0$. Thus, $cd_2(Spec(k')_{et}) = 0$. However, by the previous proposition, we have

$$cd_2(Spec(k)_{et}) \geq 1.$$

Therefore, we conclude that:

$$cd_2(Spec(k)_{et}) > cd_2(Spec(k')_{et})$$

Now, by the following proposition

Proposition 7.2.9. *Let k' be an algebraic extension of a field k, let p be a prime, and let $G_k = Gal(k^{sep}/k)$ (resp. $G_{k'} = Gal(k'^{sep}/k)$). Then $cd_p(G_{k'}) \leq cd_p(G_k)$, and there is equality if*

- *$cd_p(G_k) < +\infty$ and $[G_k : G_{k'}] < +\infty$.*

Here, the index $[G_k : G_{k'}]$ is the lcm (in the supernatural number sense) of the values of the index $[G_k, U]$ where U ranges over the open normal subgroups of G_k containing $G_{k'}$. If $|G_k|$ is finite, then this definition coincides with the classical definition, since any finite Hausdorff space is necessarily discrete.

Proof. See Proposition 10, II.4.1 of [28]. □

We have $cd_2(Spec(k)_{et}) = +\infty$ or $[G_k : G_{k'}] = \infty$. But since $[k' : k] = 2$ and $|G_k|$ is finite, by Fundamental Theorem of Galois Theory, $[G_k : G_{k'}] = 2$. Hence we obtain $cd_2(Spec(k)_{et}) = +\infty$ as desired. □

In [34], M. Artin and J.L. Verdier introduced Artin-Verdier Duality and also presented a theorem that generalizes the previous proposition.: It says that

1. An algebraic variety X over $\mathbb{R}$ has no real point if and only if $cd_2(X_{et}) < +\infty$.
2. For a filed K of finite type, k is a real field if and only if $cd_2(Spec(k)_{et})$.

Later, Claus Scheiderer extended this theorem to arbitrary schemes, using the theory he developed in his book [27].

Theorem 7.2.10. *Let X be a scheme.*

1. *If $X_r \neq \varnothing$, then $cd_2(X_{et}) = \infty$.*
2. *If X is quasi-compact and quasi-separated, and if 2 is invertible on X, and If $cd_2(X'_{et})$ is finite and the real spectrum of X is empty then also $cd_2(X_{et})$ is finite.*

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