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Introduction to Black Hole Thermodynamics

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Conclusion course paper presented to the Undergraduate Program in Physics at Instituto de Física de São Carlos, from Universidade de São Paulo, to obtain the degree of Bachelor in Physics.

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AUTORIZO A REPRODUÇÃO E DIVULGAÇÃO TOTAL OU PARCIAL DESTE TRABALHO, POR QUALQUER MEIO CONVENCIONAL OU ELETRÔNICO PARA FINS DE ESTUDO E PESQUISA, DESDE QUE CITADA A FONTE.

ABSTRACT

This work explores the mathematical and physical foundations of black hole thermodynamics within the framework of general relativity (GR). Initially, the project reviews GR's essential concepts, establishing the link between spacetime curvature and energy-momentum through Einstein's field equations. The study then transitions to specific black hole (BH) solutions, including Schwarzschild, Kerr, Reissner-Nordström and Kerr-Newmann metrics, examining their properties, event horizons and singularities. The investigation highlights the groundbreaking work by Bekenstein and Hawking, who showed that BHs possess entropy proportional to their event horizon area and can emit thermal radiation due to quantum effects. This pivotal insight integrates classical thermodynamics with quantum field theory, suggesting that BHs are thermodynamic entities rather than mere absorptive objects. The analysis includes the behavior of BHs under various cosmological conditions, focusing on their mass evolution and potential phase transitions, where heat capacity changes sign, indicating stability or instability. The results underscore the importance of BHs in connecting general relativity, thermodynamics and quantum mechanics, offering a deeper understanding of fundamental physics and potential pathways to a unified theory.

Keywords: General Relativity. Schwarzschild Black Hole. Kerr Black Hole. Reissner-Nordström Black Hole. Kerr-Newmann Black Hole. Penrose Process. Black Hole Thermodynamics. Bekenstein Entropy. Hawking Radiation. Quantum Mechanics. Heat Capacity. Cosmic Microwave Background Radiation. Mass Evolution. Phase Transitions. Thermodynamics.

1. Introduction

General relativity (GR), introduced by Albert Einstein in 1915, transformed our understanding of gravity by describing it as a curvature of spacetime rather than a conventional force. This theory successfully accounts for phenomena such as gravitational waves, light bending, and the existence of black holes (BHs), marking it as a cornerstone of modern physics. BHs emerge as solutions to Einstein's field equations and are characterized by an event horizon—a boundary beyond which no information or matter can escape. Specific solutions to these equations, like the Schwarzschild and Kerr metrics, illustrate the complex gravitational influence and singularities that these exotic objects impose upon spacetime.

Incorporating thermodynamic principles into the study of BHs led to profound theoretical advancements by linking gravitational theory with thermodynamics and quantum mechanics. Jacob Bekenstein's seminal proposal that BHs possess entropy proportional to their event horizon area suggested a deep connection to the second law of thermodynamics. Later, Stephen Hawking's discovery of BH radiation in 1974 supported this theory by showing that BHs emit thermal radiation, thereby possessing a temperature and even losing mass over time. This significant shift suggested that BHs should not be seen as mere absorptive gravitational sinks but rather as thermodynamic systems with intricate dynamics governed by laws resembling classical thermodynamics.

The motivations for this work are rooted in extending the standard undergraduate study of GR. BHs, while complex in behavior, present an elegant simplicity through the no-hair theorem, which states that their entire identity is reducible to mass, charge, and angular momentum. Despite their simplicity, BHs pose a unique complexity: understanding them introduces paradoxes and challenges to classical views on entropy and energy conservation. These puzzles led to groundbreaking discoveries that unify GR, thermodynamics, and quantum physics, a union that remains a prime motivation in physics for pursuing a fully unified theory. This project delves into the field of BH thermodynamics, covering foundational concepts, specific heat, phase transitions, and the evolutionary mass behavior of BHs in varied cosmological contexts, ultimately emphasizing their role as a bridge between macro- and micro-physical theories.

2. General Relativity

General Relativity (GR) describes gravity as the curvature of spacetime, represented by a 4-dimensional differentiable manifold \mathcal{E} . Utilizing tensor calculus, Einstein's equations link energy and momentum distributions to spacetime curvature.

2.1 Minkowski and Flat Spacetime

In Special Relativity, spacetime is modeled by Minkowski space \mathbb{M} , where the invariant interval is given by $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$. This flat space forms the foundation for extending to curved spacetime in GR.

2.2 The Metric Tensor

In GR, the **metric tensor** g_{ab} extends the Minkowski metric to curved spacetime $ds^2 = g_{ab} dx^a dx^b$. The metric tensor is symmetric ($g_{ab} = g_{ba}$) and non-degenerate, allowing the classification of vectors into **timelike**, **spacelike**, or **null**.

2.3 Einstein's Field Equations

Einstein's field equations relate spacetime curvature, represented by the Ricci tensor R_{ab} and scalar R , to the energy-momentum tensor T_{ab} :

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi G}{c^4}T_{ab}.$$

As Wheeler says: "*Matter tells spacetime how to curve, and spacetime tells matter how to move.*" (1)

2.4 Schwarzschild Solution

The Schwarzschild solution describes the spacetime geometry outside a spherically symmetric, non-rotating mass in vacuum:

$$ds^2 = - \left(1 - \frac{2GM}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

valid for $r > 2GM$, where $2GM$ is the Schwarzschild radius defining the event horizon. For simplicity, we have set $c = 1$, a convention that will often be applied throughout the text where applicable.

The **Schwarzschild BH (SBH)** is a non-rotating, uncharged BH. It is characterized by a spherical event horizon at $r = 2GM$ (where $g_{rr} \rightarrow \infty$), known as the Schwarzschild radius. This boundary defines the point at which nothing can leave once crossed.

Within the event horizon, all causal paths are directed inward, meaning that any matter or radiation inside inevitably moves toward the singularity at $r = 0$. This singularity is a region of infinite curvature and marks where the known laws of physics break down.

3. General Black Holes

3.1 The Black Hole Zoo and the No-Hair Theorem

Birkhoff's theorem asserts that spherically symmetric solutions to GR are static and asymptotically flat. When spherical symmetry is broken, gravitational fields become complex. However, BHs exhibit remarkable simplicity due to the **no-hair theorem**, which states:

Stationary, asymptotically flat BH solutions with only gravitational and electromagnetic fields are characterized solely by mass M , electric/magnetic charge Q , and angular momentum J .

Thus, regardless of initial complexity, a BH simplifies to this minimal set of parameters after reaching equilibrium.

3.2 Event Horizons and Cosmic Censorship

A BH's most defining feature is its **event horizon**, a boundary from which nothing can escape. The event horizon can be identified in stationary spacetimes by the condition $g^{rr} = 0$, signifying a surface where outgoing light rays can no longer move outward. Singularities, predicted by the **Hawking-Penrose theorems**, are considered inevitable under certain conditions. To ensure singularities remain hidden from external observers, the **cosmic censorship conjecture** posits that realistic gravitational collapses only form singularities that are shielded by an event horizon, preventing the formation of observable naked singularities.

3.3 The Rotating (Kerr) Black Holes

The **Kerr metric** describes spacetime around a rotating, uncharged BH (2):

$$ds^2 = - \left(1 - \frac{2Mr}{\rho^2} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\phi^2, \quad (3.1)$$

where $a = \frac{J}{M}$, $\rho^2 = r^2 + a^2 \cos^2 \theta$, and $\Delta = r^2 - 2Mr + a^2$. For simplicity, we will also often adopt geometric units ($c = G = 1$).

3.3.1 Properties of the Kerr Metric

- **Asymptotically Flat:** Approaches flat spacetime at large distances.
- **Stationary and Axisymmetric:** Admits Killing vectors ξ_t^α and ξ_ϕ^α , signifying time and rotational symmetries.

- **Event Horizons:** Located at $r_{\pm} = M \pm \sqrt{M^2 - a^2}$. The extremal case $a = M$ marks the **extreme Kerr** state with a singular horizon at $r_+ = M$.

3.3.2 Singularities and Horizon Area

The Kerr BH (KBH) singularity is a ring-shaped region at $\rho = 0$, representing a fundamental deviation from the point-like singularity of the SBH. The event horizon area A of a KBH is given by:

$$A = 8\pi M \left(M + \sqrt{M^2 - a^2} \right).$$

3.4 Charged (Reissner-Nordström) Black Holes

The **Reissner-Nordström (RN)** metric describes a charged (with an electric charge Q and a magnetic charge P), non-rotating BH (RNBH) (3):

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2, \quad \text{with} \quad \Delta = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2}.$$

- **Case 1** - $GM^2 > Q^2 + P^2$: Two horizons exist, similar to the KBH
- **Case 2** - $GM^2 = Q^2 + P^2$: An extremal BH with a single degenerate horizon at $r = GM$.
- **Case 3** - $GM^2 < Q^2 + P^2$: A **naked singularity** appears, violating the cosmic censorship conjecture.

3.5 Rotating (Kerr) and Kerr-Newman Black Holes

It is easy to include Q and P in the Kerr metric, simply by replacing $2GMr$ with $2GMr - G(Q^2 + P^2)$; the result is the **Kerr-Newman metric**. The KN metric generalizes the Kerr solution by including electric and magnetic charges, but for simplicity, the focus often remains on the Kerr metric as it captures the main rotational effects in the absence of charges.

3.6 Ergosphere and Energy Extraction

The **ergosphere** of a KBH, bounded by the event horizon and the **stationary limit surface**, showcases frame dragging. Particles within this region co-rotate with the BH due to the spacetime's twisting effect. Energy extraction from a rotating BH is achievable via the **Penrose process**, where a particle entering the ergosphere splits into 2 fragments. One fragment falls into the BH with negative energy $E_{\text{bh}} < 0$, while the other escapes, carrying energy E_{out} greater than the original particle's energy E_{in} :

$$E_{\text{out}} = E_{\text{in}} - E_{\text{bh}}.$$

This results in an increase in the escaping particle's energy at the expense of the BH's rotational energy. Although this process reduces both the BH's mass and angular momentum, the **area theorem** ensures that the event horizon area A cannot decrease:

$$\delta A \geq 0.$$

3.6.1 Irreducible Mass and Implications

The **irreducible mass** M_{ir} is directly linked to the horizon area A and is given by:

$$M_{\text{ir}} = \left(\frac{A}{16\pi} \right)^{1/2}, \quad M^2 = M_{\text{ir}}^2 + \frac{J^2}{4M_{\text{ir}}^2}.$$

While the angular momentum J can theoretically be reduced to zero via energy extraction, the irreducible mass M_{ir} is invariant or increases, adhering to $dM_{\text{ir}} \geq 0$.

3.7 Relation with Black Hole Thermodynamics

The extraction of rotational energy and the area theorem hint at deeper thermodynamic principles. These observations suggest intriguing parallels between classical thermodynamics and BH mechanics. The energy change in a BH system follows:

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega_H \delta J,$$

where κ is the surface gravity and Ω_H the horizon's angular velocity. This equation mirrors the first law of thermodynamics, and allow us to do the following parallels:

$$dE = TdS - pdV.$$

Zeroth Law: The surface gravity κ remains constant over the horizon, analogous to uniform temperature in thermal equilibrium.

First Law: The equation for δM aligns with thermodynamic energy change, indicating how variations in mass, angular momentum, and area are interrelated.

Second Law: Just as entropy S in a closed system does not decrease, the area A of a BH's horizon is non-decreasing.

Third Law: Reaching $\kappa = 0$ (absolute zero) is unattainable for a non-extremal BH, paralleling the thermodynamic law that absolute zero is unreachable.

These parallels between BH mechanics and thermodynamics hint at profound connections, but classical GR alone does not determine the individual normalizations of S and T . The expression $S = \frac{A}{4G}$ and $T = \frac{\kappa}{2\pi}$ gains deeper significance when quantum considerations are introduced. The idea that BHs could possess a temperature led to groundbreaking insights, which we will explore further in the next chapter.

4. Introduction to Black Hole Thermodynamics

4.1 Initial Inconsistencies with Thermodynamics

Classically, BHs were seen as purely absorptive, capable of increasing mass, charge, and angular momentum by capturing matter, but unable to emit energy. This view conflicted with the **second law of thermodynamics** since a BH seemingly “hides” the entropy of any absorbed object behind its event horizon, suggesting that entropy is lost. Additionally, the **third law** appears violated, as BHs effectively act as objects at absolute zero temperature, capable only of absorbing energy while reducing complex objects to simple parameters M , Q , and L .

4.2 Bekenstein’s Proposed Solution

To address this inconsistency, Jacob Bekenstein proposed in 1972 that BHs should be assigned an entropy, S_{bh} , proportional to their event horizon area A . This idea stemmed from the area theorem, which stated that the horizon area of a BH cannot decrease in classical processes. Bekenstein reasoned that if the area A represented a form of entropy, the total entropy, including S_{bh} , would still comply with the second law:

$$\delta (S_{\text{outside}} + S_{bh}) \geq 0.$$

He suggested that:

$$S_{bh} = \eta k \frac{A}{L_{PW}^2},$$

where η is a dimensionless constant close to unity, k is Boltzmann’s constant, and L_{PW} is the Planck length. This groundbreaking idea linked the thermodynamic concept of entropy with the geometric property of the horizon area, implying that S_{bh} could be a measure of the information lost to the outside world when objects fall into a BH.

4.3 Confirmation through Hawking Radiation

The speculative nature of Bekenstein’s proposal initially drew skepticism, especially as BHs were assumed to have a temperature of absolute zero and could not radiate energy. This view changed when Stephen Hawking, applying quantum field theory in a curved spacetime, discovered in 1974 that BHs emit thermal radiation, now known as **Hawking radiation**. This radiation results from quantum effects near the event horizon, where particle-antiparticle pairs form next to the event horizon; one particle escapes while the other falls into the BH, leading to a net decrease in the BH’s mass.

Hawking’s findings showed that the radiation emitted by BHs follows a thermal spectrum, with a temperature given by:

$$T_{bh} = \frac{\kappa \hbar}{2\pi k},$$

where κ is the surface gravity at the horizon. This result resolved the paradox of a BH absorbing external entropy, as the radiation itself increases the entropy of the external universe, maintaining the **generalized second law**.

The discovery not only confirmed Bekenstein's proposed S_{bh} with a proportionality constant $\eta = \frac{1}{4}$, but also solidified the analogy between BHs and thermodynamic systems.

4.4 BH Mechanics and BH Thermodynamics

In mid-1973, BH thermodynamics was met with widespread skepticism. Most people preferred the framework of BH mechanics, which they formulated and summarized in "The 4 Laws of BH Mechanics". They saw the analogy between BHs and thermodynamics as formal but not deeply connected to quantum mechanics. They considered $T_{bh} = 0$ and $S_{bh} \rightarrow \infty$, viewing the horizon area as analogous to entropy but rejecting the generalized 2^o law, mistakenly believing entropy could be added without increasing the area. But now we can compare the tenets of this "black-hole mechanics" with those of BH thermodynamics:

Concept	Black-hole mechanics	Black-hole thermodynamics
entropy	A is like entropy; the physical entropy is infinite	The black-hole entropy S_{bh} is $\eta c^3 kA / Gh$
temperature	θ is like temperature; the physical temperature is zero	The black-hole temperature T_{bh} is $Gh / \eta c^3 k$
first law	$\theta dA = dMc^2 - \Phi dQ - \Omega dL$	$T_{bh} dS_{bh} = dMc^2 - \Phi dQ - \Omega dL$
second law	For one black hole A cannot decrease; when black holes coalesce the total horizon area increases (area theorem)	The sum of S_{bh} and the entropy exterior to black holes cannot decrease (generalized second law)

Table 1 – Comparison between black-hole mechanics and black-hole thermodynamics, showing concepts of entropy, temperature, and the first and second laws.

Source: Reproduced from Bekenstein (1980, Table 2) (4).

4.4.1 Black Hole Temperature

Consider the expression for S_{bh} of a Kerr-Newman BH (KNBH) (Table 2 shows some of its relations); its total differential with respect to M , Q and L can be written as

$$\theta dA = T_{bh} dS_{bh} = d(Mc^2) - \Phi dQ - \Omega dL \quad (4.1)$$

where T_{bh} is defined in terms of the non-negative quantity θ of table 1 by $T_{bh} = L_{PW}^2 \theta / \eta k$, so that ΦdQ is the work done on the hole by adding dQ to it, ΩdL is the work done by adding dL and, of course, $d(Mc^2)$ is the corresponding change in the hole's energy. This equation mirrors the 1^o law of thermodynamics, suggesting dS_{bh} is a genuine entropy and T_{bh} a quantum temperature. For the numerical value of T_{bh} as function of BH mass, see Figure 1. Note that the curves behavior turns clear as we check that $S_{bh} \sim A \sim M^2$ and $T_{bh} \propto dM/dA \sim 1/M$.

Parameter	mass M	charge Q	angular momentum L
Characteristic length	$m = GM/c^2$ roughly size of hole	$q = (GQ^2/c^4)^{1/2}$	$a = L/Mc$
Quantity equivalent to 1 cm	1.35×10^{26} g	3.49×10^{24} esu	3×10^{10} cm ² /s specific angular momentum
Constraint	$m^2 \geq q^2 + a^2$		
Formula for horizon area	$A = 4\pi[(m + (m^2 - q^2 - a^2)^{1/2})^2 + a^2]$		
Potential	surface tension θ	electric potential Φ	angular velocity Ω
Definition	$(\partial Mc^2/\partial A)_{L,Q}$	$(\partial Mc^2/\partial Q)_{L,A}$	$(\partial Mc^2/\partial L)_{Q,A}$
Formula	$(m^2 - q^2 - a^2)^{1/2} c^4 / 2GA$	$Q[4\pi/A - (4\pi a/A)^2]^{1/2}$	$(L/M)4\pi/A$
	$c^4/2G = 6.07 \times 10^{48}$ erg/cm		

Table 2 – The KNBH parameters, including mass M , charge Q , and angular momentum L .

Source: Reproduced from Bekenstein (1980, Table 1) (4).

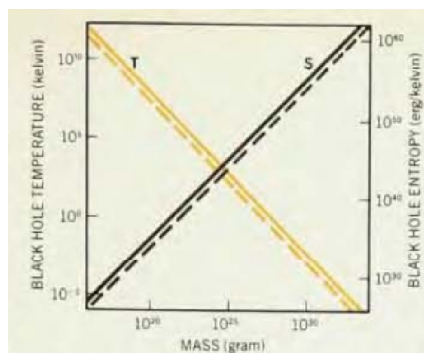


Figure 1 – Graph showing the relationship between black hole mass, temperature (T), and entropy (S). As the mass decreases, the temperature increases while the entropy decreases.

Source: Reproduced from Bekenstein (1980, Fig. 5) (4).

Hawking’s findings also resolved the paradox of BHs in thermal equilibrium with their environment, as the radiation ensures that the generalized second law holds. This established T_{bh} as a genuine physical temperature and S_{bh} as a true entropy, providing critical evidence for the connection between thermodynamics and quantum mechanics. An important consequence observed in this relationship is an indication of SBH instability. As an SBH gains energy (M), it lowers its T due to the inverse relationship $T_{bh} \propto 1/M$, making it more challenging to reach thermal equilibrium. This runaway behavior suggests that an SBH cannot stabilize within an environment at a fixed temperature.

4.5 Broader Implications and Further Developments

Hawking’s work demonstrated that BH entropy is not merely theoretical but has practical implications for understanding the quantum nature of gravitation. This breakthrough showed that BHs could be studied as quantum systems with defined thermodynamic properties, offering insights into phenomena such as the **information paradox** and the ultimate fate of evaporating BHs. The thermal nature of BH radiation hinted at deeper principles linking gravity and quantum theory, as seen in related phenomena like the Unruh effect and de Sitter space radiation.

5. Differential Relations in BH Thermodynamics

The analogy between BH physics and thermodynamics extends beyond conceptual parallels into mathematical expressions. This chapter broadens the analogy by introducing partial differential relations for BHs that mirror standard thermodynamic equations. Here, since the focus is primarily on analyzing mathematical relations, we will disregard constants by using geometric units.

5.1 Analogous Thermodynamics

The combined first and second laws of classical thermodynamics are:

$$dU = TdS - PdV,$$

where U is internal energy, T is temperature, S is entropy, P is pressure, and V is volume. In BH thermodynamics, the analogous equation becomes:

$$dM = \theta d\alpha + \vec{\Omega} \cdot d\vec{L} + \Phi dQ,$$

with $\alpha = \frac{A}{4\pi}$, $\theta = \frac{1}{4} \frac{r_+ - r_-}{\alpha}$, $\vec{\Omega} = \frac{\vec{a}}{\alpha}$, and $\Phi = \frac{Qr_+}{\alpha}$. In this context, $\vec{\Omega} \cdot d\vec{L}$ and ΦdQ represent the work terms due to spin and electric field, respectively. The "temperature" θ is defined by:

$$\theta = \left(\frac{\partial M}{\partial \alpha} \right)_{L, Q}.$$

Analogous to thermodynamic potentials, enthalpy H_{bh} , Helmholtz free energy F_{bh} , and Gibbs free energy G_{bh} can be defined for BHs:

$$dH_{bh} = \theta d\alpha - \vec{L} \cdot d\vec{\Omega} - Qd\Phi,$$

$$dF_{bh} = -\alpha d\theta + \vec{\Omega} \cdot d\vec{L} + \Phi dQ,$$

$$dG_{bh} = -\alpha d\theta - \vec{L} \cdot d\vec{\Omega} - Qd\Phi.$$

These potentials allow the derivation of partial differential equations similar to those in classical thermodynamics.

5.2 Maxwell's Relations and Differential Equations

Maxwell's relations are fundamental in classical thermodynamics, connecting various partial derivatives. For BHs, these relations also apply, providing insights into how different variables interact.

For a Kerr-Newman BH, complex interactions due to multiple work terms make deriving these equations challenging. But, from $d\alpha = \left(dM - \vec{\Omega} \cdot d\vec{L} - \Phi dQ \right) / \theta$, we can establish relations between partial derivatives:

Thermodynamics	Kerr–Newman black hole	
	Kerr black hole	Reissner–Nordström black hole
$\left(\frac{\partial S}{\partial P}\right)_V = -\left(\frac{\partial V}{\partial T}\right)_S$	$(\nabla_{\mathbf{a}}\alpha)_{\mathbf{L}} = \left(\frac{\partial \mathbf{L}}{\partial \theta}\right)_{\alpha}$	$\left(\frac{\partial \alpha}{\partial \Phi}\right)_{\varrho} = \left(\frac{\partial Q}{\partial \theta}\right)_{\alpha}$
$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$	$(\nabla_{\mathbf{L}}\alpha)_{\theta} = -\left(\frac{\partial \Omega}{\partial \theta}\right)_{\mathbf{L}}$	$\left(\frac{\partial \alpha}{\partial Q}\right)_{\theta} = -\left(\frac{\partial \Phi}{\partial \theta}\right)_{\varrho}$
$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$	$(\nabla_{\mathbf{a}}\alpha)_{\theta} = \left(\frac{\partial \mathbf{L}}{\partial \theta}\right)_{\mathbf{a}}$	$\left(\frac{\partial \alpha}{\partial \Phi}\right)_{\theta} = \left(\frac{\partial Q}{\partial \theta}\right)_{\Phi}$
$\left(\frac{\partial S}{\partial V}\right)_P = \left(\frac{\partial P}{\partial T}\right)_S$	$(\nabla_{\mathbf{L}}\alpha)_{\theta} = -\left(\frac{\partial \Omega}{\partial \theta}\right)_{\alpha}$	$\left(\frac{\partial \alpha}{\partial Q}\right)_{\Phi} = -\left(\frac{\partial \Phi}{\partial \theta}\right)_{\alpha}$
	$\nabla_{\mathbf{a}} \equiv \frac{\partial}{\partial \Omega_x} \hat{i} + \frac{\partial}{\partial \Omega_y} \hat{j} + \frac{\partial}{\partial \Omega_z} \hat{k}$	

Table 3 – Maxwell equations and their analogous forms for BHs.

Source: Reproduced from Parker (1980, Table 3) (5).

5.3 Specific Heat and Stability

Specific heat, a critical concept in thermodynamics, applies to BHs as well. It is defined as:

$$C_K = T \left(\frac{\partial S}{\partial T} \right)_K,$$

where K denotes constant parameters. For SBHs, the specific heat C_S is (in geometric units) :

$$C_S = -\frac{1}{T^2},$$

indicating that, being always negative, they get hotter as they radiate energy, leading to eventual instability; just as predicted in the previous chapter. But now, for KBHs with constant Ω , the specific heat C_{Ω} is (in geometric units) :

$$C_{\Omega} = T \left(\frac{\partial S}{\partial T} \right)_{\Omega} = -\frac{2TSM^{-1}}{16\Omega^2 + T^2},$$

and for RNBHs with constant Φ (in geometric units) :

$$C_{\Phi} = T \left(\frac{\partial S}{\partial T} \right)_{\Phi} = -2S.$$

These expressions show how BHs behave under various conditions and their tendency towards stability or instability based on specific parameters.

6. Mass Evolution in Schwarzschild Black Holes

The negative heat capacity of SBHs (found in the previous chapter) implies they cannot reach thermal equilibrium. This chapter explores the evolution of BH mass over time, considering various scenarios such as an SBH in empty space and within the cosmic microwave background radiation (CMB) with both constant and evolving temperatures through cosmic eras.

6.1 The Negative Heat Capacity

The entropy S and temperature T of an SBH are given by:

$$T = \frac{1}{8\pi M}, \quad S = 4\pi M^2,$$

where M is the BH's mass and $A = 16\pi M^2$ is the event horizon's area. The heat capacity C is:

$$C = \frac{dE}{dT} = \frac{dM}{dT} = -\frac{1}{8\pi T^2} = -8\pi M^2 < 0.$$

This negative heat capacity means that SBHs deviate from thermal equilibrium. If $T > T_R$ (the reservoir temperature), the BH emits radiation and its mass decreases while its temperature increases. Conversely, if $T < T_R$, the BH absorbs energy, gains mass, and its temperature decreases.

6.2 Mass Evolution in Empty Space

For a BH in empty space with $T_R = 0$, the energy exchange leads to continuous radiation and mass loss. The emitted power L , derived from black body radiation, is (in SI):

$$L = \frac{\hbar c^6}{15360\pi G^2} \frac{1}{M^2},$$

which results in the mass evolution equation:

$$\frac{dM}{dt} = -\frac{\hbar c^6}{15360\pi G^2} \frac{1}{M^2}.$$

Solving this yields:

$$M(t) = M_0 \left(1 - \frac{t}{t_v}\right)^{1/3},$$

where $t_v = 5120\pi \frac{G^2}{\hbar c^4} M_0^3$ is the BH's lifetime. The behavior is depicted in Fig. 2, showing that the BH eventually evaporates completely.

6.3 Mass Evolution with a Constant CMB

When an SBH is immersed in the CMB with $T_{CMB} \approx 2.7255$ K, the energy exchange follows:

$$J = \sigma A(T^4 - T_{CMB}^4),$$

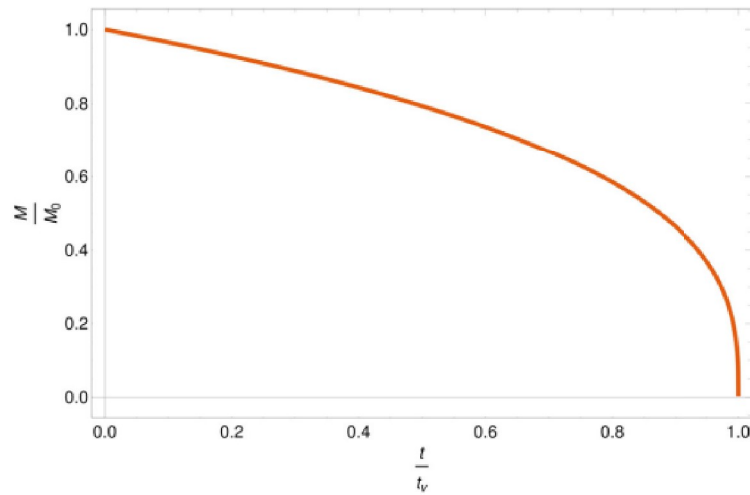


Figure 2 – General mass decay of an SBH in empty space.

Source: Reproduced from Natali (2019, Fig. 1) (6).

yielding the differential equation:

$$\frac{dM}{dt} = -\frac{a}{M^2} + bM^2,$$

where $a \approx 4.0 \times 10^{15} \text{ kg}^3 \text{ s}^{-1}$ and $b \approx 9.6 \times 10^{-76} \text{ s}^{-1} \text{ kg}^{-1}$. The first term represents mass loss due to Hawking radiation, and the second term indicates mass gain from CMB absorption.

The solution behavior depends on the initial mass M_0 :

- **For** $M < M_{CMB}$: The first term dominates, leading to evaporation (mass decreases to zero).
- **For** $M > M_{CMB}$: The second term dominates, resulting in mass growth (indefinite increase).

The stream plot in Fig. 3 highlights these behaviors.

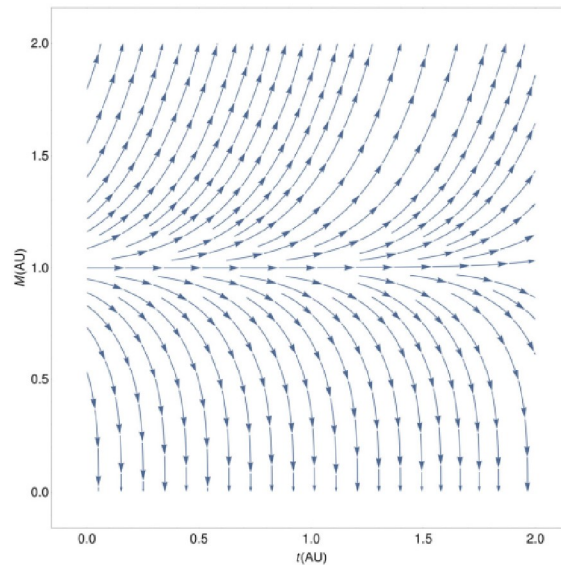


Figure 3 – Stream plot showing mass evolution of an SBH in a constant CMB temperature.

Source: Reproduced from Natali (2019, Fig. 2) (6).

6.4 Mass Evolution in a Time-Evolving CMB

The universe's thermal history influences SBH mass evolution, as CMB temperature evolves through the radiation, matter, and dark energy eras. The temperature behavior is:

$$T_R(t) \sim t^{-1/2} \text{ (radiation era)}, \quad T_M(t) \sim t^{-2/3} \text{ (matter era)}, \quad T_\Lambda(t) \sim e^{-a_\Lambda t} \text{ (dark energy era)}.$$

The mass evolution equations are:

$$\text{Radiation era: } \frac{dM}{dt} = -\frac{a}{M^2} + ba_R \frac{M^2}{t^2}, \quad (6.1)$$

$$\text{Matter era: } \frac{dM}{dt} = -\frac{a}{M^2} + ba_M \frac{M^2}{t^{8/3}}, \quad (6.2)$$

$$\text{Dark energy era: } \frac{dM}{dt} = -\frac{a}{M^2} + bM^2 e^{-a_\Lambda t}. \quad (6.3)$$

Here, $a = \frac{h^4}{15360\pi G^2} \approx 4.0 \times 10^{15} \text{ kg}^3 \text{ s}^{-1}$, $b = \frac{4\pi^3 k_B^4 G^2 T_{CMB}^4}{15h^3 c^8} \approx 9.6 \times 10^{-76} \text{ s}^{-1} \cdot \text{kg}^{-1}$,
 $a_R = \left(\frac{32\pi G}{3}\rho_{R0}\right)^{-1}$, $a_M = (6\pi G\rho_{M0})^{-4/3}$, $a_\Lambda = \left(\frac{128\pi G}{3}\rho_{\Lambda0}\right)^{1/2}$.

For each era:

- **First term dominance:** The BH emits energy and its mass decreases.
- **Second term dominance:** The BH absorbs energy and its mass increases.

In Fig. 4, we have constructed the stream plot for the differential Eq. 6.1 to represent the general behavior for all the solutions of Eqs. 6.1-6.3. The behavior of these differential equations shows 2 possible scenarios: the mass will decrease to zero (evaporation) or increase (divergence) depending on its initial mass and universe era.

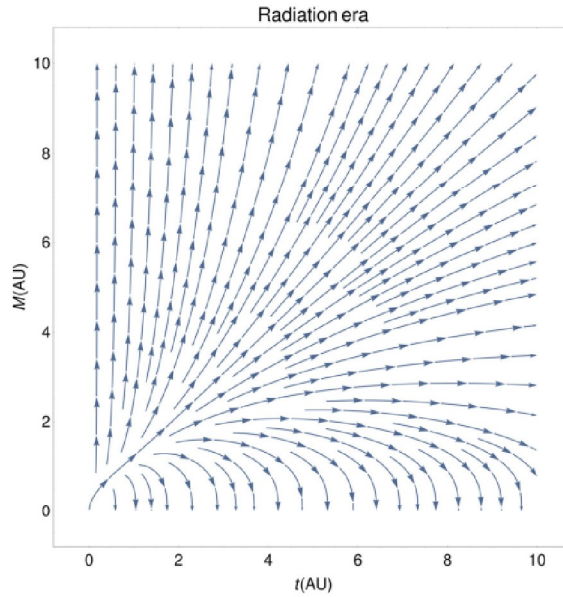


Figure 4 – Stream plot (mass versus time) for a SBH exchanging energy with CBM in the radiation era.

Source: Reproduced from Natali (2019, Fig. 5) (6).

7. Phase Transitions in Black Hole Thermodynamics

In BH thermodynamics, phase transitions reveal distinctive stability properties, especially in KNBHs. While SBHs possess negative heat capacity, causing them to run away from equilibrium as they radiate energy, KNBHs can achieve a positive heat capacity through a phase transition. This transition enables KNBHs to potentially reach thermal equilibrium, contrasting with the inherent instability of SBHs. This chapter explores the critical points of such transitions, where thermodynamic potentials, like specific heat, display discontinuities that underscore the parallels between BH mechanics and classical thermodynamics. Also, we going to use geometric units in this chapter too.

7.1 Basic Relations for Kerr-Newmann Black Hole

The thermodynamic relations for KNBHs express (7):

$$M^2 = \frac{1}{4} \left(\frac{A}{4\pi} \right) + \frac{4\pi}{A} \left(J^2 + \frac{1}{4} Q^4 \right) + \frac{1}{2} Q^2. \quad (7.1)$$

Since S is related to A , and considering that BHs cannot be divided into subsystems, we have:

$$M(S, J, Q) = \sqrt{2S + \left(\frac{1}{8S} \right) \left(J^2 + \frac{1}{4} Q^4 \right) + \frac{1}{2} Q^2}. \quad (7.2)$$

The first law of thermodynamics for KNBHs is formulated as:

$$dM = TdS + \Omega dJ + \Phi dQ, \quad (7.3)$$

where the BH's temperature T , angular velocity Ω , and electric potential Φ are given by:

$$T = \frac{\partial M}{\partial S} = M^{-1} \left[1 - \frac{J^2 + \frac{1}{4} Q^4}{16S^2} \right],$$

$$\Omega = \frac{\partial M}{\partial J} = \frac{J}{8MS},$$

$$\Phi = \frac{\partial M}{\partial Q} = \frac{Q(Q^2 + 8S)}{16MS}.$$

Applying Euler's theorem for homogeneous functions gives:

$$\frac{1}{2}M = TS + \Omega J + \frac{1}{2}\Phi Q = TS + \Omega J + \Theta Q^2, \quad \text{where: } \Theta = \frac{\Phi}{2Q} = \frac{Q^2 + 8S}{32MS}.$$

Which is analogous to the Gibbs-Duhem relation in thermodynamics. From this, we obtain:

$$\left(\frac{\partial M}{\partial T} \right)_{\Omega, \Theta} = -2S, \quad \left(\frac{\partial M}{\partial \Omega} \right)_{T, \Theta} = -2J, \quad \left(\frac{\partial M}{\partial \Theta} \right)_{T, \Omega} = -2Q^2.$$

7.2 Phase Transition, Heat Capacity and Gibbs Free Energy

With the relations provided in the previous section, the specific heat at constant angular momentum J and charge Q for a KNBH may be expressed as:

$$C_{J,Q} = T \left(\frac{\partial S}{\partial T} \right)_{J,Q} = \frac{8MS^3T}{J^2 + \frac{1}{4}Q^4 - 8T^2S^3}. \quad (7.4)$$

For BHs, phase transitions occur when $a = \frac{J}{M} = 0.68M$. For RNBHs, the phase transition is at $Q = 0.86M$. These points indicate an infinite discontinuity in $C_{J,Q}$, marking a second-order phase transition where the heat capacity switches from negative to positive.

The Gibbs free energy for KNBHs is defined by $G = M - TS - \Omega J - \Phi Q$. Although G remains continuous across the phase transition, its second derivatives (such as heat capacity and compressibility) show discontinuities, which align with classical definitions of 2^o-order phase transitions.

7.3 Graphical Analysis of Heat Capacity

The general behavior of $C_{J,Q}$ is depicted in Fig. 5, showing the infinite discontinuity at the phase transition point. This behavior highlights the region where the heat capacity shifts from negative (unstable) to positive (potentially stable). The broken line in the figure indicates this transition, which is crucial for understanding the thermodynamic stability of KNBHs.

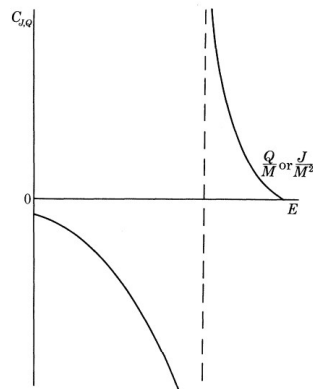


Figure 5 – General behavior of $C_{J,Q}$. For low values of $\frac{J}{M^2}$ and $\frac{Q}{M}$, the $C_{J,Q} < 0$, indicating instability. The broken line marks the phase transition where $C_{J,Q}$ becomes infinite. Beyond this line, $C_{J,Q}$ is positive, suggesting a stable configuration until it drops to zero at the extreme Kerr–Newman limit E .

Source: Reproduced from Davies (1977, Fig. 1) (7).

For a KBH with $Q = 0$ at the critical point we have the relation $\Omega/T \simeq 0.23$, which defines a straight phase line dividing regions of positive and negative specific heat in Fig. 6.

In the RN limit, where $J = 0$, the phase transition occurs at a constant electric potential $\Phi = 1/\sqrt{3}$, indicating that this transition is independent of the BH's mass.

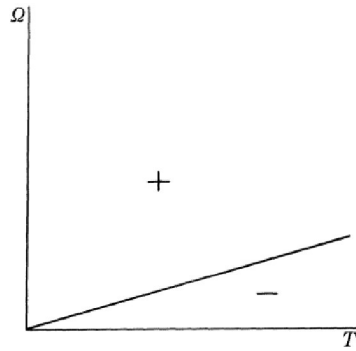


Figure 6 – Phase diagram for KBH. The + and – phases correspond to positive and negative values of the specific heat at constant J . The dividing phase line is $\Omega = 0.23T$.

Source: Reproduced from Davies (1977, Fig. 2) (7).

7.4 Isentropes and Isotherms

Figures 7 and 8 show the isentropes and isotherms for RNBHs and KBHs. These figures provide insight into how S and T change across different charge and angular momentum values.

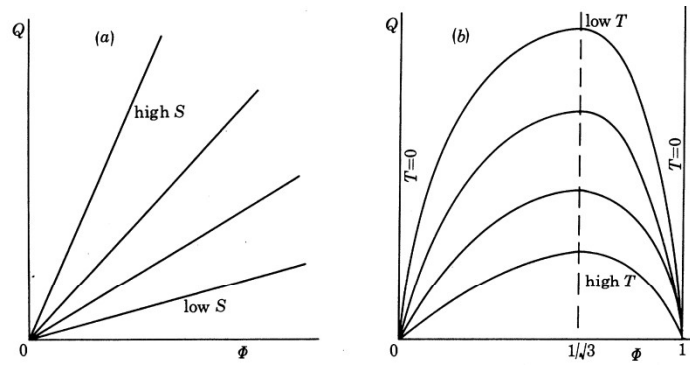


Figure 7 – (a) Isentropes for RNBHs. (b) Isotherms for RNBHs. The phase transition occurs at $\Phi = \frac{1}{\sqrt{3}}$. Each Q, T combination typically corresponds to two possible values of Φ .

Source: Reproduced from Davies (1977, Fig. 3) (7).

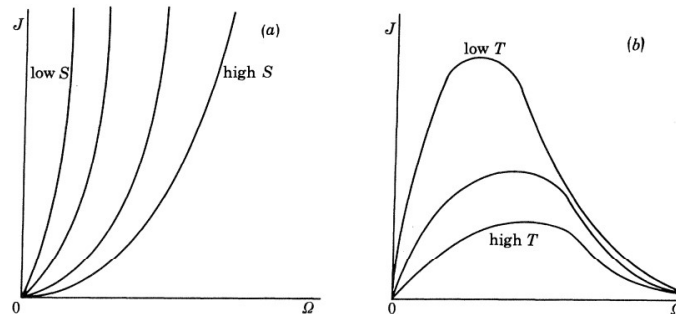


Figure 8 – (a) Isentropes for KBHs. (b) Isotherms for KBHs. Each J, T combination generally corresponds to two values of Ω . This diagram helps visualize how the S and T evolve with varying J .

Source: Reproduced from Davies (1977, Fig. 4) (7).

8. Conclusion

Black holes, with their fundamental link to the fabric of spacetime and thermodynamics, offer a compelling study in modern physics. Although they involve complex Einstein equations, the tools used here reveal that much of BH thermodynamics can be explored with standard undergraduate methods, with only minimal reliance on advanced mathematics. This study spanned critical topics, including BH entropy, energy extraction, Hawking's resolution of entropy paradoxes, and the mass evolution of BHs under cosmic conditions. By understanding these aspects, we uncovered the influence of heat capacity on BH stability, its dependence on mass, angular momentum, and charge, and the conditions for equilibrium in specific cases.

Through this journey, we found that BHs are more than gravitational oddities; they embody a significant early connection between GR, quantum mechanics, and thermodynamics. Kerr-Newman BHs, for instance, exhibit phase transitions allowing for stability in thermal equilibrium, contrasting with Schwarzschild BHs, which diverge away from equilibrium. Hawking's and Bekenstein's work on BH thermodynamics not only reconciled longstanding theoretical issues but also paved the way for a broader understanding of quantum phenomena in extreme gravitational environments. By advancing the field toward a unified physical theory, BH thermodynamics has become a gateway, offering promising pathways in the search for an all-encompassing framework that reconciles GR and quantum mechanics.

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